1. (13 pts) A particle is moving in such a way that it's acceleration is given by $\mathbf{a}(t) = \langle 4, \sin(t), e^t \rangle$. The initial velocity is $\mathbf{v}(0) = \langle -6, 2, 0 \rangle$.

(a) (5 pts) Find the curvature, κ, at time $t = 0$.

$\vec{r}'(0) = \langle -6, 2, 0 \rangle$

$\vec{r}''(0) = \langle 4, 0, 1 \rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -6 & 2 & 0 \\ 4 & 0 & 1 \end{vmatrix} = \langle 2-0, -(-6-0), 0-8 \rangle$$

$$\kappa(0) = \frac{|\vec{r}'(0) \times \vec{r}''(0)|}{|\vec{r}'(0)|^3} = \frac{\sqrt{4+36+64}}{(36+4+0)^{3/2}} = \boxed{\frac{\sqrt{104}}{40^{3/2}} \approx 0.040311}$$

(b) (8 pts) Assume the particle starts at $\mathbf{r}(0) = \langle 0, 2, 3 \rangle$ (so it starts on the yz-plane). The particle will pass through the yz-plane again at some later time. Find the (x, y, z) coordinates at which the particle passes through the yz-plane again. (Hint: First find $\mathbf{r}(t)$).

$\vec{v}(t) = \langle 4t + c_1, -\cos(t) + c_2, e^t + c_3 \rangle$

$\vec{v}(0) = \langle -6, 2, 0 \rangle \Rightarrow c_1 = -6, \quad -1 + c_2 = 2, \quad 1 + c_3 = 0$

$c_2 = 3, \quad c_3 = -1$

$\vec{r}(t) = \langle 2t^2 - 6t + d_1, -\sin(t) + 3t + d_2, e^t - t + d_3 \rangle$

$\vec{r}(0) = \langle 0, 2, 3 \rangle \Rightarrow d_1 = 0, \quad d_2 = 2, \quad 1 + d_3 = 3$

$d_3 = 2$

$2t^2 - 6t = 0 \Rightarrow 2t(t-3) = 0 \Rightarrow t = 3$

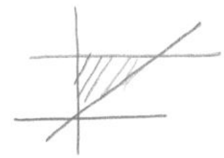
$\vec{r}(3) = \langle 0, -\sin(3) + 3(3) + 2, e^3 - 3 + 2 \rangle$

$= \boxed{\langle 0, -\sin(3) + 11, e^3 - 1 \rangle}$

7

2. (a) (7 pts) Set up and evaluate a double integral to find the volume of the solid below the surface $z - 3x^2y + 1 = 0$, above the surface $z = 1$ and between the planes $x = 0$, $y = 2$, and $y = 2x$.

$$\int_0^1\int_{2x}^2 3x^2y + \cancel{1} - \cancel{1}\,dy\,dx$$



$$\int_0^1 \left(\frac{3}{2}x^2y^2\Big|_{2x}^{2}\right)dx$$

$$\int_0^1 \frac{3}{2}x^2(2)^2 - \frac{3}{2}x^2(2x)^2\,dx$$

$$\int_0^1 6x^2 - 6x^4\,dx$$

$$2x^3 - \frac{6}{5}x^5\Big|_0^1$$

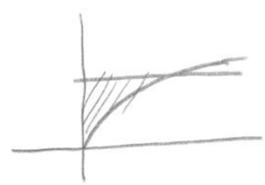
$$2 - 6/5 = \boxed{4/5}$$

Also could do

$$\int_0^2\int_0^{\frac{1}{2}y} 3x^2y\,dx\,dy$$

(b) (7 pts) Evaluate ~~the integral by reversing the order of integration~~: $\displaystyle\int_0^4\int_{\sqrt{x}}^2 \frac{x}{y^5+1}\,dy\,dx.$

$$\int_0^2\int_0^{y^2}\frac{x}{y^5+1}\,dx\,dy$$



$$\int_0^2 \frac{1}{y^5+1}\left(\frac{1}{2}x^2\Big|_0^{y^2}\right)dy$$

$$\frac{1}{2}\int_0^2 \frac{y^4}{y^5+1}\,dy$$

$u = y^5 + 1$

$du = 5y^4\,dy$

$$\frac{1}{10}\int_1^{33}\frac{1}{u}\,du$$

$$\frac{1}{10}\left(\ln|u|\Big|_1^{33}\right) = \boxed{\frac{1}{10}\ln(33)}$$

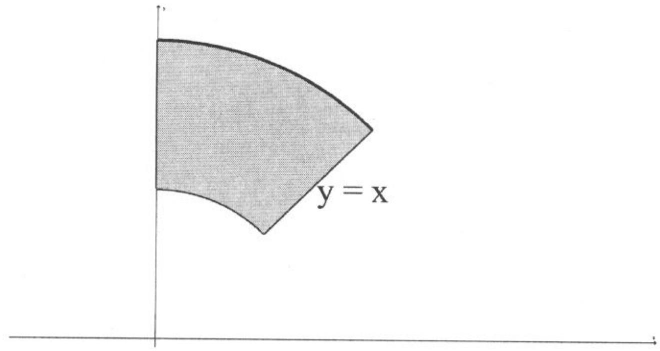
3. (10 pts) A lamina occupies the region R in the first quadrant that is above the line $y = x$ and between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ (as shown below). The density is proportional to the distance from the origin.

Find the y-coordinate of the center of mass, $\bar{y}$. (Give your final answer as a decimal to 4 digits).

$\pi/4 \le \theta \le \pi/2$

$1 \le r \le 2$

$\rho(x,y) = k\sqrt{x^2+y^2}$



$$M = \text{TOTAL MASS} = \int_{\pi/4}^{\pi/2}\int_1^2 kr \, r\,dr\,d\theta = k\int_{\pi/4}^{\pi/2} d\theta \int_1^2 r^2\,dr$$
$$= k\left(\tfrac{\pi}{2} - \tfrac{\pi}{4}\right)\left(\tfrac{1}{3}r^3\Big|_1^2\right)$$
$$= \frac{k\pi}{4}\left(\frac{8}{3} - \frac{1}{3}\right)$$
$$= \frac{7k\pi}{12}$$

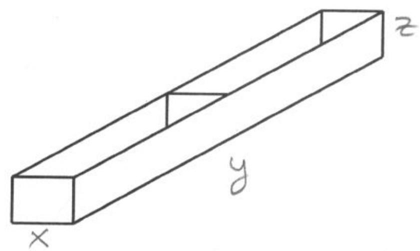
$$m_x = \int_{\pi/4}^{\pi/2}\int_1^2 r\sin(\theta)\, kr\, r\,dr\,d\theta$$
$$= k\int_{\pi/4}^{\pi/4} \sin(\theta)\,d\theta \int_1^2 r^3\,dr$$
$$= k\left(-\cos\theta \Big|_{\pi/4}^{\pi/2}\right)\left(\tfrac{1}{4}r^4\Big|_1^2\right)$$
$$= k\left(-0 - -\tfrac{\sqrt{2}}{2}\right)\left(4 - \tfrac{1}{4}\right)$$
$$= \frac{k\sqrt{2}}{2}\cdot\frac{15}{4} = \frac{15k\sqrt{2}}{8}$$

$$\bar{y} = \frac{m_x}{M} = \frac{15\sqrt{2}}{8}\cdot\frac{12}{7\pi} = \boxed{\frac{45\sqrt{2}}{14\pi} \approx 1.4469}$$

4. (13 pts)You are designing a cage to hold your pet rabbits. The cage is a rectangular box with a bottom, four sides, and one divider in the middle (you are keeping the males and females apart). There is no top. A picture of such a cage is below.

If you want the total combined volume to be 12 cubic feet, then what dimensions will minimize the total material used?

Give your final dimensions as decimals to four digits. As part of your answer, **use the second derivative test** to verify that your critical point is a local minimum.



CONSTRAINT: $xyz = 12 \Rightarrow z = \frac{12}{xy}$

OBJECTIVE:

MATERIAL $= xy + 2yz + 3xz$

$f(x,y) = xy + \frac{24}{x} + \frac{36}{y}$

$f_x = y - \frac{24}{x^2} \overset{?}{=} 0 \Rightarrow y = \frac{24}{x^2}$

$f_y = x - \frac{36}{y^2} \overset{?}{=} 0 \Rightarrow x = \frac{36}{y^2}$

Combine and solve $\Rightarrow x = \frac{36}{(24/x^2)^2} \Rightarrow x = \frac{36}{24^2}x^4 = \frac{1}{16}x^4$

$\Rightarrow 16 = x^3 \Rightarrow x = 16^{1/3}$

$y = \frac{24}{x^2} = \frac{24}{16^{2/3}}$

$z = \frac{12}{xy} = \frac{12}{24/16^{1/3}} = \frac{1}{2}16^{1/3}$

$f_{xx} = \frac{48}{x^3} = \frac{48}{16} = 3$

$f_{yy} = \frac{72}{y^3} = \frac{4}{3} = 1.\overline{3}$

$f_{xy} = 1$

$D = 3 \cdot \frac{4}{3} - 1^2 = 3$

$D > 0$, $f_{xx} > 0 \Rightarrow$ LOCAL MIN

$(x,y,z) = (2.5198, 3.7798, 1.2599)$