Math 126 Exam 1 Commentary

The solutions to exam 1 are on the following pages. But first read this.

Suggestions for Going Over This Exam:

1. Spend a couple minutes looking through the solutions for ALL the problems (for the ones you got right and the ones you got wrong). You might see a better way to do a problem.

2. Get out your study materials for exam 1 (the old exams you studied, the homework you studied, the notes from when you studied). Review what you studied and compare to what you missed.

3. Briefly as you look at this exam, you should see that nearly everything in this exam can be found in many places in the homework and old exams:

   (a) Page 1: A standard page about lines and planes.
   (b) Page 2: Naming a surface. Problem 2(b) required you think about intersection of surfaces. Problem 3 was a standard type of polar problem.
   (c) Page 3: Parametric Equations in 2D.
   (d) Page 4: A curve in 3D. Tangent vector, tangent line, and angle of intersection.

General comments about how we grade:

1. The grading is consistent! Page 1 was graded by the same TA for ALL students in my Math 126 courses. Page 2 was graded by another TA for ALL students, etc... In addition, I gave all the graders detailed grading guides. So the grading is consistent and well thought out.

2. Our job is to assess your ability to show your understanding on the test and to gauge your likelihood of success in subsequent courses. We can only grade what you put on the test. Some students try to submit a long explanation of what they 'meant' in hopes to get more credit. We don't give extra credit and I won't grade extra work that you write up now, we grade what you actually put on the test.

3. Major algebra mistakes or fundamental arthimetic mistakes (that show dramatic misunderstanding of material that preceeded this course) result in big deductions as they indicate it will be difficult for you to succeed beyond this course.

4. Major conceptual mistakes result in very big deductions as they show that you don't understand the current material.

5. We do give partial credit for showing understanding even if the final answer is wrong. However, we do not guarantee partial credit for everything you write down. For example, just copying a formula from a notesheet does not necessarily get you any credit. And we reserve the right to take off full points in certain situations where the magnitude of the mistakes outweighs anything that might be considered worth partial credit.

6. If a student makes a small calculation mistake, we typically don't take off very many points, unless that small mistake dramatically changes the problem making it much simpler than intended (in which case we must take off more points since we didn't get to see all the necessary work for the problem).

7. If it is clear that the method you show is completely wrong, but you coincidentally get an answer near the correct answer you do NOT get any points (you don't get points for pure luck).

1. (12 points)

(a) Find the equation of the plane that goes through the three points $\overset{A}{(1,2,5)}$, $\overset{B}{(2,2,2)}$, $\overset{C}{(3,3,3)}$.

$\overrightarrow{AB} = \langle 1, 0, -3 \rangle$

$\overrightarrow{AC} = \langle 2, 1, -2 \rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -3 \\ 2 & 1 & -2 \end{vmatrix} = (0 - {-3})\vec{i} - (-2 - -6)\vec{j} + (1 - 0)\vec{k}$$

$= \langle 3, -4, 1 \rangle$ ← check dot products ✓

NORMAL

$$\boxed{\begin{gathered} 3(x-1) - 4(y-2) + (z-5) = 0 \\ \text{ANY POINT} \\ 3x - 3 - 4y + 8 + z - 5 = 0 \\ 3x - 4y + z = 0 \end{gathered}}$$

(b) Find parametric equations for the line of intersection of $x - z = 10$ ① and $x + y + 2z = 0$ ②.

FIND TWO POINTS : $x = 0 \Rightarrow z = -10$ in ①

① & ② $\Rightarrow (0) + y + 2(-10) = 0 \Rightarrow y = 20$

$P(0, 20, -10)$

$z = 0 \Rightarrow x = 10$ in ①

① & ② $\Rightarrow (10) + y + 2(0) = 0 \Rightarrow y = -10$

$Q(10, -10, 0)$

DIRECTION : $\vec{v} = \overrightarrow{PQ} = \langle 10, -30, 10 \rangle$

$$\boxed{\begin{aligned} x &= 0 + 10t \\ y &= 20 - 30t \\ z &= -10 + 10t \end{aligned}}$$

ANY POINT ON LINE

ANY VECTOR PARALLEL TO $\langle 10, -30, 10 \rangle$

2. (5 pts) Consider the surface $z^2 - 6x^2 - 6y^2 = -9$.

(a) (2 pts) Give the precise name of this surface as given in the book and in class.

multiply by -1 to get $-z^2 + 6x^2 + 6y^2 = 9$ → also traces hyp/hyp/circle ← always defined

$\boxed{\text{HYPERBOLOID OF ONE SHEET}}$

(b) (3 pts) The surfaces $z^2 - 6x^2 - 6y^2 = -9$ and $z = x^2 + y^2$ intersect to form a circle that is parallel to the xy-plane. Find the center and radius of this circle.
(For the center, give the (x, y, z) coordinates).

$z^2 - 6(x^2+y^2) = -9$ AND $x^2+y^2 = z$ combine to give

$z^2 - 6z = -9 \Rightarrow z^2 - 6z + 9 = 0 \Rightarrow (z-3)^2 = 0 \Rightarrow \boxed{z=3}$

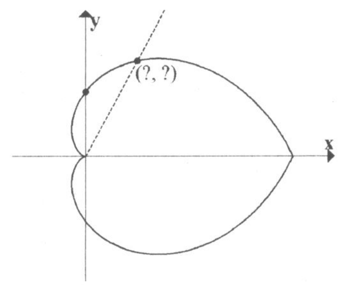
$x^2 + y^2 = 3$

CENTER : $(0,0,3)$

RADIUS : $\sqrt{3}$

3. (8 pts) Consider the polar curve $r = 1 - \sin\left(\frac{1}{2}\theta\right)$. (shown below)

(a) (3 pts) In the first quadrant, the line $y = \sqrt{3}x$ intersects the curve at the origin and at one other point (as shown). Find this other point.
(Note: The line makes a 60 degree angle with the positive x-axis).



$\theta = \pi/3 \qquad r = 1 - \sin\left(\frac{1}{2}\frac{\pi}{3}\right) = 1 - \frac{1}{2} = 1/2$

$x = \frac{1}{2}\cos(\pi/3) = \frac{1}{2}\,\frac{1}{2} = 1/4$

$y = \frac{1}{2}\sin(\pi/3) = \frac{1}{2}\,\frac{\sqrt{3}}{2} = \sqrt{3}/4$

(b) (5 pts) The polar curve has one positive y-intercept. Find the slope of the tangent line at this point.

$\theta = \pi/2 \qquad r(\pi/2) = 1 - \sin\left(\frac{1}{2}\frac{\pi}{2}\right) = 1 - \frac{\sqrt{2}}{2}$

$\frac{dr}{d\theta} = -\frac{1}{2}\cos\left(\frac{1}{2}\theta\right) \Rightarrow \left.\frac{dr}{d\theta}\right|_{\theta=\pi/2} = -\frac{1}{2}\cos\left(\frac{1}{2}\frac{\pi}{2}\right) = -\frac{\sqrt{2}}{4}$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{dr/d\theta\,\sin\theta + r\cos\theta}{dr/d\theta\,\cos\theta - r\sin\theta} \stackrel{\theta=\pi/2}{=} \frac{(-\frac{\sqrt{2}}{4})(1) + (1-\frac{\sqrt{2}}{2})(0)}{(-\frac{\sqrt{2}}{4})(0) - (1-\frac{\sqrt{2}}{2})(1)}$

$= \boxed{\frac{-\sqrt{2}/4}{-1+\sqrt{2}/2} = \frac{-\sqrt{2}}{-4+2\sqrt{2}}}$

6 - 2

4. (10 pts) Consider the parametric equations $x = t^2 - 3t$, $y = 12t - t^3$ (curve shown below).

(a) Find the (x, y) coordinates of all locations on the curve at which the tangent line is horizontal.

WANT $\frac{dy}{dx} \stackrel{?}{=} 0 \iff \frac{dy/dt}{dx/dt} = \frac{12-3t^2}{2t-3} \stackrel{?}{=} 0$

$12 - 3t^2 = 0 \Rightarrow t^2 = 4 \Rightarrow t = \pm 2$

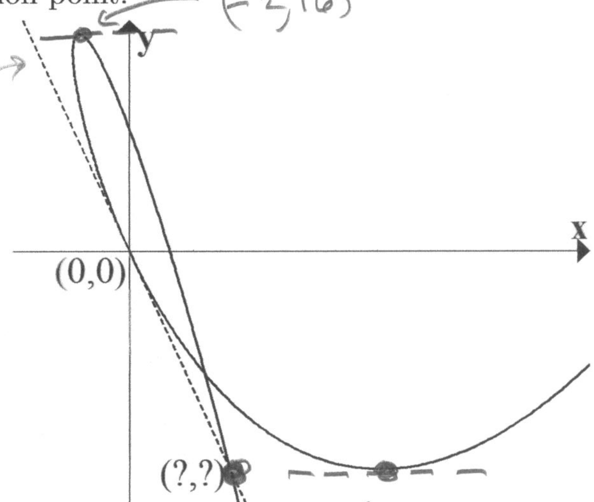
$t = -2 \Rightarrow x = (-2)^2 - 3(-2) = 10, \; y = 12(-2) - (-2)^3 = -16$

$t = 2 \Rightarrow x = (2)^2 - 3(2) = -2, \; y = 12(2) - 2^3 = 16$

$\boxed{(10, -16) \quad \text{AND} \quad (-2, 16)}$

(b) The tangent line at $t = 0$ also intersects the curve at one other point (as shown). Find the (x, y) coordinates of the other intersection point.

(-2,16)



(10,-16)

At $t = 0$, $x = 0, y = 0$, and

$\frac{dy}{dx} = \frac{12 - 0}{0 - 3} = -4$

Thus, $y = -4(x - 0) + 0$

$y = -4x$

INTERSECTION:

$12t - t^3 = -4(t^2 - 3t)$

$\cancel{12t} - t^3 = -4t^2 + \cancel{12t}$

$0 = t^3 - 4t^2$

$0 = t^2(t - 4)$ $\begin{cases} t = 0 \text{ or} \\ t = 4 \end{cases}$

$t = 4 \Rightarrow x = (4)^2 - 3(4) = 4, \; y = 12(4) - (4)^3 = -16$

$\boxed{(4, -16)}$

5. (15 pts) At time $t = 0$, an egg is thrown up into the air toward Dr. Loveless by a disgruntled student. The egg's path is described parametrically by $\underbrace{x = t,\ y = 8\sqrt{t+1},\ z = 15t - t^2}_{\vec{r}_1(t)}$.

(a) (3 pts) At the time $t = 0$, find a vector that is tangent to the curve and has length 5.

$\vec{r}_1\,'(t) = \langle 1, \frac{4}{\sqrt{t+1}}, 15 - 2t \rangle$, $\vec{r}_1\,'(0) = \langle 1, 4, 15 \rangle$

$|\vec{r}_1\,'(0)| = \sqrt{1+16+225} = \sqrt{242} = 11\sqrt{2}$

+ or − okay

$$\boxed{\frac{5}{\sqrt{242}} \langle 1, 4, 15 \rangle = \left\langle \frac{5}{\sqrt{242}}, \frac{20}{\sqrt{242}}, \frac{75}{\sqrt{242}} \right\rangle}$$

(b) Find parametric equations for the tangent line at the positive time when the egg hits the xy-plane.

HITS xy-PLANE $\Leftrightarrow$ $z = 15t - t^2 \overset{?}{=} 0 \Leftrightarrow t(15-t) = 0$

$t = 0$ or $t = 15$

$\vec{r}_1(15) = \langle 15, 8\sqrt{16}, 0 \rangle = \langle 15, 32, 0 \rangle$

$\vec{r}_1\,'(15) = \langle 1, \frac{4}{\sqrt{16}}, 15 - 2(15) \rangle = \langle 1, 1, -15 \rangle$

$$\boxed{\begin{aligned} x &= 15 + t \\ y &= 32 + t \\ z &= 0 - 15t \end{aligned}}$$

(c) A rock is also flying through the air following the path $\overbrace{x = 3,\ y = 14 + u,\ z = 28 + u^3}^{\vec{r}_2(u)}$. The path of the rock and the path of the egg intersect (unfortunately for Dr. Loveless, the rock and egg don't collide). Find the (acute) angle of intersection of the two paths. Give your final answer in degrees rounded to two digits after the decimal.

INTERSECTION: $t \overset{?}{=} x \overset{?}{=} 3$ $\quad t = 3$

$8\sqrt{t+1} \overset{?}{=} y \overset{?}{=} 14 + u \Rightarrow 16 = 14 + u \Rightarrow u = 2$

$15t - t^2 \overset{?}{=} z \overset{?}{=} 28 + u^3 \Rightarrow$ CHECK $\underbrace{15(3) - (3)^2}_{36} \overset{?}{=} \underbrace{28 + (2)^3}_{36}$ ✓

$\vec{r}_1\,'(3) = \langle 1, 2, 9 \rangle = \vec{u}$

$\vec{r}_2\,'(u) = \langle 0, 1, 3u^2 \rangle$ $\quad \vec{r}_2\,'(2) = \langle 0, 1, 12 \rangle = \vec{v}$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{0 + 2 + 108}{\sqrt{1+4+81}\sqrt{0+1+144}} = \frac{110}{\sqrt{86}\sqrt{145}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{110}{\sqrt{86}\sqrt{145}}\right) \approx 9.92 \text{ degrees}}$$