Practice Examples

The final exam is comprehensive. You will be expected to know the definitions and methods discussed in lecture, homework, and previous exams. The questions below are all similar to homework examples. These example should not be your only source of study material.

1. **Mathematical Statements**
   (a) Give definitions of all of the following: Converse, Contrapositive, Proof by Contradiction, Tautology
   (b) Suppose that P and Q are true statements and R and S are false statements. Which of the following are given a truth value of TRUE?
      i. $R \Rightarrow P$.
      ii. (P or R) and S.
      iii. $Q \Rightarrow (P \Rightarrow \neg S)$
      iv. $\neg$(R or Q) $\Leftrightarrow S$.
   (c) Use a truth table to show that the statement $\neg P \Rightarrow \neg Q$ is equivalent to the statement $Q \Rightarrow P$.

2. **Set Problems** Let A, B, and C be sets.
   (a) Prove that $(A \cup B) - C \subseteq (A - C) \cup B$. And give an example where $(A \cup B) - C \neq (A - C) \cup B$
   (b) Prove that $A \cup (A \cap B) = A$.
   (c) Prove that $(A \cup B) \cap A^c = B - A$.
   (d) Recall: For a function $f : A \to B$, we define
   $f(S) = \{y \in B : y = f(x) \text{ for some } x \in S\}$ where $S \subseteq A$. Let $S, T \subseteq A$.
      i. Prove that $f(S \cap T) \subseteq f(S) \cap f(T)$. And give an example where these sets are not equal.
      ii. Prove that if f is an injection, then $f(S \cap T) = f(S) \cap f(T)$.

3. **Induction Problems**
   (a) Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$.
   (b) Prove that $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$ for all $n \in \mathbb{N}$.
   (c) Prove that $5^n + 5 < 5^{n+1}$ for all $n \in \mathbb{N}$.
   (d) Prove that if $1 + x > 0$, then $(1 + x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

4. **Function Problems**
   (a) If the function $f : [0, \infty) \to [0, 1)$ given by $f(x) = \frac{x}{x+1}$ injective? Prove your answer.
   (b) For all of the following give a function $h : \mathbb{Z} \to \mathbb{Z}$ that satisfy the given properties:
      i. h is injective but not surjective.
      ii. h is surjective but not injective.
      iii. h is a bijection.
   (c) Give proofs for the true statements and counterexamples for the false statements:
      i. Every unbounded function from $\mathbb{R}$ to $\mathbb{R}$ is not a sujection.

ii. Every surjective function from $\mathbb{R}$ to $\mathbb{R}$ is montone.
iii. Every monotone function from $\mathbb{R}$ to $\mathbb{R}$ is injective.
iv. Every strictly monotone function from $\mathbb{R}$ to $\mathbb{R}$ is a bijection.

(d) Let $f : A \to B$, $g : B \to C$ and $h = g \circ f$. Prove or give a counterexample for the following:
i. If h is injective, then f is injective.
ii. If h is injective, then g is injective.
iii. If h is sujective, then f is surjective.
iv. If h is surjective, then g is surjective.
v. If g and f are bijections, then h is a bijection.

(e) Cardinality Stuff: Review the homework.

5. **Divisibility Problems**

(a) If n is divisible by 2 and m is not divisible by 2, then $m + n$ is not divisible by 2.

(b) If a and b are odd integers, then $a^2 - b^2$ is divisible by 8.

(c) If a is an odd integer, then $a^2 + (a + 2)^2 + (a + 4)^2 + 1$ is divisible by 12.

(d) If $n = a^2 + b^2$, then n cannot be of the form $n = 4m + 3$ for any $m \in \mathbb{Z}$.

(e) Let m and n be any positive integers. Show that if m and n have no common *prime* factor, then $m + n$ and m have no common prime factor. (Hint: For practice, try proving the contrapositive.)

(f) Use the Euclidean Algorithm to compute the greatest common divisor of 234 and 366.

(g) Give all integer solutions to $234x + 366y = 126$.

6. **Binomial Coefficients**

(a) What is the value of $B = \sum_{k=0}^{11} \binom{11}{k} 7^k (-9)^{11-k}$? Give a formula, in terms of n, that gives the value of $B = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} w^k (w + 1)^{n-k}$.

(b) Using a result from homework, Prove that if $\binom{n-2}{k}$ and $\binom{n-2}{k-2}$ are both odd, then $\binom{n}{k}$ is even.

(c) Use the binomial theorem to prove, $(m + 1)^k - 1$ is always divisible by m for integers $k \geq 0$.

7. **Congruences**

(a) Let $n \in \mathbb{N}$. Prove that 7 divides $6^n + 1$ if and only if n is odd.

(b) Solve $5x \equiv 3 \pmod{18}$ by finding a multiplicative inverse of 5 modulo 18 and multiplying both sides.

(c) Find the remainder when 18200000014743273^4 is divided by 10.

(d) Prove that if $bc \equiv bd \pmod{p}$, where p is a prime such that p does not divide b, then $c \equiv d \pmod{p}$.

(e) Let n be a positive integer. Prove that 3 divides n if and only if 3 divides the sum of the base 10 digits of n.

(f) Prove that if $ab \equiv 0 \pmod{n}$ and $b \not\equiv 0 \pmod{n}$, then $a \equiv 0 \pmod{n}$ or $\gcd(b, n) > 1$.

(g) Prove that if $\gcd(a^2 + a + 1, n) = 1$ and $a^3 \equiv 1 \pmod{n}$, then $a \equiv 1 \pmod{n}$.
Give an example where $\gcd(a^2 + a + 1, n) > 1$ and $a \not\equiv 1 \pmod{n}$.