1. (a) Give a counterexample to each following statement:

   i. Every one-to-one function from $\mathbb{R}$ to $\mathbb{R}$ is onto.
   ANSWER: $f(x) = e^x$ is one-to-one and not onto from $\mathbb{R}$ to $\mathbb{R}$.
   (Another example is $f(x) = \arctan(x)$) $\square$

   ii. Every unbounded function from $\mathbb{R}$ to $\mathbb{R}$ is monotone.
   ANSWER: $f(x) = x^2$ is unbounded and not monotone from $\mathbb{R}$ to $\mathbb{R}$.
   (Other examples are $f(x) = x^4$ and $f(x) = x^3 - x$) $\square$

   PROBLEM NOTES: Most people got full credit on this problem.

   - **$f(x) = \frac{1}{x}$ is not function FROM $\mathbb{R}$ to $\mathbb{R}$. That is, it is not defined on all of $\mathbb{R}$, so it is doesn't work as a counterexample to (i) or (ii).**
   - **$f(x) = x^3 + x$ is not a counterexample to (ii), it is always increasing to the second part.**
   - **On (ii), most people used $f(x) = x^3 - x$, but $f(x) = x^2$ is a simpler example.**

(b) Give the coefficient of x^{19} in the expansion of $(x + 2)^{21}$.
ANSWER:

$$\binom{21}{19} = \frac{21!}{19!2!} = \frac{21 \cdot 20}{2 \cdot 1} = 21 \cdot 10 = 210$$

The term with x^{19} would look like $210x^{19}2^2$. So the coefficient of x^{19} is

$$210 \cdot 4 = 840. \qquad \square$$

PROBLEM NOTES: Most people got full credit on this problem as well (it was just like a problem on the Fall 2008 Exam 2).

- **A couple people forgot about the 2.**

(c) Let $g : \mathbb{R} - \{0\} \to \mathbb{R}$ by $g(x) = 5 + \frac{3}{x}$.
Prove g is injective.

PROOF: Assume $g(x_1) = g(x_2)$ for some x_1, x_2 in $\mathbb{R} - \{0\}$.
Then

$$\begin{aligned}
5 + \tfrac{3}{x_1} &= 5 + \tfrac{3}{x_2} && \textbf{(from the definition of } g\textbf{)} \\
5x_1x_2 + 3x_2 &= 5x_1x_2 + 3x_1 && \textbf{(multiplying both sides by } x_1x_2\textbf{)} \\
3x_2 &= 3x_1 && \textbf{(canceling } 5x_1x_2\textbf{)} \\
x_2 &= x_1 && \textbf{(canceling 3)}
\end{aligned}$$

Thus, g is one-to-one. $\square$

PROBLEM NOTES: Almost everyone got this problem. It turned out to be the easiest problem on the page.

2. (a) Use the Euclidean algorithm to compute $\gcd(208, 84)$.
(You MUST show me the steps of the Euclidean algorithm to get full credit)
ANSWER:

$$\begin{aligned} 208 &= (2)84 + (40) \\ 84 &= (2)40 + (4) \\ 40 &= (10)4 + (0) \end{aligned}$$

Thus, $\gcd(208, 84) = 4$. $\square$

(b) Does $208x + 84y = 15$ have a solution for $x, y \in \mathbb{Z}$? If so, find the solution for x and y that is given by Euclidean algorithm. If not, explain why.
ANSWER: $\gcd(208, 84) = 4$ **DOES NOT DIVIDE 15. By the LDE Theorem, there is no integer solution.** $\square$

(c) Does $208x + 84y = 20$ has a solution for $x, y \in \mathbb{Z}$? If so, find the solution for x and y that is given by the Euclidean algorithm. If not, explain why.
ANSWER: $\gcd(208, 84) = 4$ **DOES DIVIDE 20. Back solving in the Euclidean algorithm gives**

$$\begin{aligned} 4 &= 84 - (2)40 \\ 4 &= 84 - (2)(208 - (2)84) \\ 4 &= 84 - (2)208 + (4)84 \\ 4 &= 208(-2) + 84(5) \end{aligned}$$

Thus, one solution to $208x_0 + 84y_0 = 4$ **is** $x_0 = -2$ **and** $y_0 = 5$.
To get a solution for $208x + 84y = 20$ **we multiply by 5 to get** $208(5x_0) + 84(5y_0) = 20$ **which gives the solution** $x = -10$ **and** $y = 25$. $\square$

PROBLEM NOTES: Scores were good ((a) was worth 6, (b) was 2, (c) was 4).

- **In (c), I said you need to find the solution for** x **and** y **that is given by the Euclidean algorithm. You needed to back solve (and then multiply by 5).**

(d) Prove that if 3 divides a and 3 divides b, then 9 divides $6b + (a + 1)^3 - 1$.
PROOF: Since 3 divides a **and 3 divides** b**, there exists integers** m **and** n **such that** $a = 3m$ **and** $b = 3n$.
Then

$$\begin{aligned} 6b + (a+1)^3 - 1 &= 6b + a^3 + 3a^2 + 3a + 1 - 1 && \textbf{(by the binomial theorem)} \\ &= 6(3m) + (3n)^3 + 3(3n)^2 + 3(3n) && \textbf{(by substitution)} \\ &= 18m + 27n^3 + 27n^2 + 9n && \textbf{(simplification)} \\ &= 9(2m + 3n^3 + 3n^2 + n) && \textbf{(factoring)} \end{aligned}$$

Thus, 9 divides $6b + (a + 1)^3 - 1$. $\square$

PROBLEM NOTES: The scores were very good on this problem.

- **If you were stumped, you should have started with the definition of divides.**
- **I did NOT say that** $\gcd(a, b) = 3$. **In fact, from class, we can say that 3 divides** $\gcd(a, b)$**, but we can't say** $3 = \gcd(a, b)$**, so the LDE theorem does not apply (don't use this theorem if the** gcd **function isn't in the problem).**
- **Some of you multiplied** $(a + 1)^3$ **out the long way. Use the binomial theorem, it makes your work faster.**

3. Define $G_0 = 1$, $G_1 = 1$ and $G_n = 1 + 3G_{n-1} - 2G_{n-2}$ for $n \geq 2$.
Using strong induction, prove that $G_n = 2^n - n$ for all integers $n \geq 0$.

PROOF:

Base Step:
For $n = 0$, $G_0 = 1$, by given, and $2^n - n = 2^0 - 0 = 1$.
For $n = 1$, $G_1 = 1$, by given, and $2^n - n = 2^1 - 1 = 1$.

Inductive Step: Assume $G_i = 2^i - i$ for $i = 0, 1, \ldots, k$ for some $k \geq 1$.

Using the recurrence relation gives

$$\begin{aligned}
G_{k+1} &= 1 + 3G_k - 2G_{k-1} && \textbf{(by the given definition)} \\
&= 1 + 3(2^k - k) - 2(2^{k-1} - (k-1)) && \textbf{(by the ind. hyp. twice)} \\
&= 1 + 3 \cdot 2^k - 3k - 2^k + 2k - 2 && \textbf{(expanding and noting } 2 \cdot 2^{k-1} = 2^k\textbf{)} \\
&= 2 \cdot 2^k - k - 1 && \textbf{(combining and simplifying)} \\
&= 2^{k+1} - (k+1) && \textbf{(simplifying)}
\end{aligned}$$

Thus, $G_n = 2^n - n$ for all integers $n \geq 0$. $\square$

PROBLEM NOTES: There were a couple common mistakes on this problem. There were too many who didn't know the structure of a strong induction proof. Please look back at my Ch. 3 review (bottom of the first and top of the second page). *When you prepare for a test in this class your first job is to know the structure of the types of proofs we have been doing!*

- **Some students did not lay out the format for a strong induction proof. You need to correctly state the strong inductive hypothesis (some stated the regular inductive hypothesis, which wouldn't work because you needed to replace G_{k-1} during the induction). In strong induction, we assume the equality holds for $i = 0, 1, \ldots k$ and we prove it holds for $i = k + 1$ (we never gave a form of induction that only assume k and $k - 1$).**
- **Some of you are too easily distracted by the definition of G_n. You are not proving the definition of G_n. The statement you are proving is $G_n = 2^n - n$ for all integers $n \geq 0$. So your base step is $n = 0$ (and you also have to do $n = 1$ so that you can use the recurrence formula in the inductive step).**
- **Four or five students made serious algebra mistakes. $3 \cdot 2^k$ is NOT the same as 6^k and $2 \cdot 2^k$ is NOT the same as 4^k. Be careful with your algebra, in courses after this one, instructors will expect that you are very comfortable with algebra.**

4. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function and define the function $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = 2x - f(x)$ for all $x \in \mathbb{R}$.

Prove that if f is nonincreasing, then h is increasing.

PROOF: Let x_1, x_2 be in $\mathbb{R}$ such that $x_1 < x_2$.
Since f is nonincreasing, we have $f(x_1) \geq f(x_2)$.
Multiplying by (-1) gives $-f(x_1) \leq -f(x_2)$.
Adding $2x_1$ to both sides gives $2x_1 + [-f(x_1)] \leq 2x_1 + [-f(x_2)]$. Also since $x_1 < x_2$, we have $2x_1 < 2x_2$. Thus,

$$2x_1 - f(x_1) \leq 2x_1 - f(x_2) < 2x_2 - f(x_2).$$

Hence, $h(x_1) < h(x_2)$ and, therefore, h is increasing. $\square$

PROBLEM NOTES: Most of you did very well on this problem.

- **You should have known to start with $x_1 < x_2$. A few had the wrong definition of nonincreasing.**
- **If you add a strict inequality to a nonstrict inequality, the result is a strict inequality (for example, adding $3 < 4$ and $5 \leq 5$ gives $8 < 9$).**
- **A few of you missed the fact that the inequality $f(x_1) \geq f(x_2)$ flips when multiplied by -1.**

(b) Let $f : A \to B$ and $g : B \to C$ be functions and define the function $h : A \to C$ by $h(x) = g(f(x))$ for all $x \in A$. Prove that if h is onto and g is one-to-one, then f is onto.

PROOF: Let $b \in B$.
Since g is defined from B to C, $g(b) = c$ for some $c \in C$
(*i.e.* c is the particular output that came from b).
Since h is onto, $h(a) = c$ for some $a \in A$.
By definition of h, $g(f(a)) = h(a) = c$.
Hence, $g(b) = c = g(f(a))$ and since g is one-to-one, $b = f(a)$ for some $a \in A$. $\square$

PROBLEM NOTES: Far too many of you didn't know how to start a proof that f is onto. Please look back at my Ch. 4 review (midway down the first page). *When you prepare for a test in this class your first job is to know the structure of the types of proofs we have been doing!*

- **If you didn't start with $b \in B$, then I should have given you a zero on the problem because you don't know the definition of onto. But I did find ways to give partial credit. The goal of an onto proof is to show that given any element from the target of the function (in this case B), that there is an element in A that maps to be.**
- **Thus, after you start with a particular $b \in B$, you explain (using this particular b), why you can get $f(a) = b$. A few of you started with $b \in B$, but then the symbol b never appear again in your proof (this does not make sense).**
- **Some of you were completely ignoring the sets. I saw a few exams where elements of C were being plugged into g (this does not make sense).**
- **A few of you started by saying "Let $b \in B$ such that $f(x) = b$ for some $x \in A$." This is assuming the conclusion and doesn't make sense. The part "... $f(x) = b$ for some $x \in A$." is what you are trying to end with.**