Math 300 Summer 2010 Exam 2

Name: ____

1	18	
2	22	
3	20	
4	20	
Total	80	

- USE YOUR TIME WISELY! START EACH PROBLEM BY OUTLINING THE PROOFS/IDEAS ON THE PAGE. THE LAST TWO PAGES ARE PROOFS. SPEND NO MORE THAN 10 MINUTES ON EACH PAGE.
- Put your answers on the exam, there are 4 pages (you may use your own paper for scratch work). If you need more space for your answer use the back of the preceding page and indicate that you have done so.
- If you don't know how to prove something, then you should still let me know everything you know about the problem (that is, give definitions and facts you know about the problem and discuss different techniques to prove the theorem). Also, if you are stuck you should try some examples. Don't leave any question blank, give me the opportunity to give you credit for what you know!

GOOD LUCK!

1. (a) Give a counterexample to each following statements:

   i. (4 pts) Every nonincreasing function from $\mathbb{R}$ to $\mathbb{R}$ is injective.

   ii. (4 pts) Every surjective function from $\mathbb{R}$ to $\mathbb{R}$ is monotone.

   (b) (4 pts) Let $A = \{x : \cos(x) = 0 \text{ and } x \in \mathbb{R}\}$.
   Is the set A finite, countably infinite, or uncountably infinite? And very briefly explain why.

   (c) (6 pts) Using the binomial theorem and Pascal's triangle to help you expand, prove that for all $x, y \in \mathbb{Z}$, the number $(x+y)^5 - x^5 - y^5$ is divisible by 5.

2. (a) Recall that $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0\}$.

   i. (8 pts) Giving a precise proof, show that $f(x) = 4x$ is a bijection from $\mathbb{Q}$ to $\mathbb{Q}$.

   ii. (4 pts) Explain why $f(x) = 4x$ is NOT a bijection from $\mathbb{Z}$ to $\mathbb{Q}$. Give a specific counterexample to one of the conditions required for a bijection.

   (b) (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a function.
   Define another function $g : [0, \infty) \to \mathbb{R}$ by $g(x) = f\left(\frac{x}{x+1}\right)$.
   Prove if f is injective, then g is injective.

CHECK YOUR TIME! LEAVE 20 MINUTES FOR THE LAST PAGE!

3. (a) (5 pts) **Showing all the appropriate steps of the Euclidean algorithm**, calculate the greatest common divisor of 240 and 345.

   (b) (15 pts) Consider the sequence defined by $a_1 = 3$, $a_2 = 9$, and $a_n = a_{n-1}^2 - 7a_{n-2} + 12$ for $n \geq 3$. Using the precise definition of divisibility by 3 and the precise phrasing for a strong induction proof, show that a_n is divisible by 3 for all $n \in \mathbb{N}$.

REMEMBER TO VERY CLEARLY GIVE THE **ORDER AND JUSTIFICATIONS** BELOW
Note that parts (a) and (b) are completely independent.

4. (a) (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions.
Suppose g is **nondecreasing** and f is **nonincreasing**.
Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = g(x) - f(x)$.
Which of the following must be true about h? (Circle your answer)

NONDECREASING NONINCREASING INCREASING DECREASING

NONE OF THE ABOVE

Give a full proof completely justifying your answer.

(b) (10 pts) Let $f : \mathbb{R} \to (2, \infty)$ and $g : (2, \infty) \to (0, \infty)$ be functions.
Define $h : \mathbb{R} \to (0, \infty)$ by $h(x) = 3g(f(x - 4))$.
Prove that if f is surjective and g is surjective, then h is surjective.