1. (a) Give a counterexample to each following statements:

i. (4 pts) Every nonincreasing function from $\mathbb{R}$ to $\mathbb{R}$ is injective.

$\boxed{f(x) = c \text{ where } c \text{ is a constant}}$

is nonincreasing and NOT injective from $\mathbb{R}$ to $\mathbb{R}$

ii. (4 pts) Every surjective function from $\mathbb{R}$ to $\mathbb{R}$ is monotone.

Here are a few answers.

$\boxed{f(x) = x^3 - x}$ is surjective and NOT monotone from $\mathbb{R}$ to $\mathbb{R}$

$\boxed{f(x) = x\sin(x)}$ is surjective and NOT monotone from $\mathbb{R}$ to $\mathbb{R}$

$\boxed{f(x) = \begin{cases} x & \text{if } x < -1, x > 1; \\ -x & \text{if } -1 \leq x \leq 1. \end{cases}}$ is surjective and NOT monotone from $\mathbb{R}$ to $\mathbb{R}$

(b) (4 pts) Let $A = \{x : \cos(x) = 0 \text{ and } x \in \mathbb{R}\}$.
Is the set A finite, countably infinite, or uncountably infinite? And very briefly explain why.

$A = \{-\frac{\pi}{2}, \frac{\pi}{2}, -\frac{3\pi}{2}, \frac{3\pi}{2}, -\frac{5\pi}{2}, \frac{5\pi}{2}, \ldots\}$

$\boxed{A \text{ is countably infinite}}$

The elements of A are listed in correspondence with $\mathbb{N}$.

In particular, $f(x) = \begin{cases} -\frac{x\pi}{2}, & x \text{ is odd}; \\ \frac{(x-1)\pi}{2}, & x \text{ is even} \end{cases}$ is

a bijection from $\mathbb{N}$ to A.

(c) (6 pts) Using the binomial theorem and Pascal's triangle to help you expand, prove that for all $x, y \in \mathbb{Z}$, the number $(x+y)^5 - x^5 - y^5$ is divisible by 5.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

pf/ Let $x, y \in \mathbb{Z}$.
By the binomial theorem

$$(x+y)^5 - x^5 - y^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 - x^5 - y^5$$
$$= 5\underbrace{(x^4y + 2x^3y^2 + 2x^2y^3 + xy^4)}$$
$$= 5m$$

where $m = x^4y + 2x^3y^2 + 2x^2y^3 + xy^4 \in \mathbb{Z}$

Hence, 5 divides $(x+y)^5 - x^5 - y^5$. //

2. (a) Recall that $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0\}$.

i. (8 pts) Giving a precise proof, show that $f(x) = 4x$ is a bijection from $\mathbb{Q}$ to $\mathbb{Q}$.

$\boxed{1-1}$ Let $f(x_1) = f(x_2)$ with $x_1, x_2 \in \mathbb{Q}$.
Then $4x_1 = 4x_2$, so $x_1 = x_2$ (dividing by 4).

$\boxed{\text{ONTO}}$ Let $y \in \mathbb{Q}$, so $y = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$, $b \neq 0$.
$4x = y$ is equivalent to $x = y/4 = \frac{a}{4b}$.
So $f(x) = y$ with $x = \frac{a}{4b} \in \mathbb{Q}$. //

ii. (4 pts) Explain why $f(x) = 4x$ is NOT a bijection from $\mathbb{Z}$ to $\mathbb{Q}$. Give a specific counterexample to one of the conditions required for a bijection.

$4x = 1$ has no sol'n in $\mathbb{Z}$.
$1 \in \mathbb{Q}$. $1/4 \notin \mathbb{Z}$.

There are many other examples

(b) (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a function.
Define another function $g : [0, \infty) \to \mathbb{R}$ by $g(x) = f\left(\frac{x}{x+1}\right)$.
Prove if f is injective, then g is injective.

pf) Let $g(x_1) = g(x_2)$ for $x_1, x_2 \in [0, \infty)$.
By defn of g, $f\left(\frac{x_1}{x_1+1}\right) = f\left(\frac{x_2}{x_2+1}\right)$.
Since f is one-to-one, $\frac{x_1}{x_1+1} = \frac{x_2}{x_2+1}$.
By multiplying by $(x_1+1)(x_2+1)$ and simplifying,

$$x_1(x_2+1) = x_2(x_1+1)$$
$$\cancel{x_1x_2} + x_1 = \cancel{x_2x_1} + x_2$$
$$x_1 = x_2 \; //$$

CHECK YOUR TIME! LEAVE 20 MINUTES FOR THE LAST PAGE!

3. (a) (5 pts) **Showing all the appropriate steps of the Euclidean algorithm,** calculate the greatest common divisor of 240 and 345.

$345 = 1 \cdot 240 + 105$
$240 = 2 \cdot 105 + 30$
$105 = 3 \cdot 30 + \boxed{15}$
$30 = 2 \cdot 15$

$\boxed{\gcd(345, 240) = 15}$

(b) (15 pts) Consider the sequence defined by $a_1 = 3$, $a_2 = 9$, and $a_n = a_{n-1}^2 - 7a_{n-2} + 12$ for $n \geq 3$. Using the precise definition of divisibility by 3 and the precise phrasing for a strong induction proof, show that a_n is divisible by 3 for all $n \in \mathbb{N}$.

pf]

BASE STEP For $n=1$: $a_1 = 3$, by def'n, and $3 = 3(1)$, so $3 \mid a_1$.

For $n=2$: $a_2 = 9$, by def'n, and $9 = 3(3)$, so $3 \mid a_2$.

IND STEP Assume 3 divides a_i for $i = 1, 2, \ldots, k$ for some $k \geq 2$.

By def'n,

$a_{k+1} = a_k^2 - 7a_{k-1} + 12.$

By the ind hyp., $a_k = 3m_1$ and $a_{k-1} = 3m_2$ for some $m_1, m_2 \in \mathbb{Z}$. By substitution,

$a_{k+1} = (3m_1)^2 - 7(3m_2) + 12 = 3(3m_1^2 - 7m_2 + 4).$

Hence, 3 divides a_{k+1}. //

REMEMBER TO VERY CLEARLY GIVE THE **ORDER AND JUSTIFICATIONS** BELOW
Note that parts (a) and (b) are completely independent.

4. (a) (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions.
Suppose g is **nondecreasing** and f is **nonincreasing.**
Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = g(x) - f(x)$.
Which of the following must be true about h? (Circle your answer)
NONDECREASING NONINCREASING INCREASING DECREASING
NONE OF THE ABOVE
Give a full proof completely justifying your answer.

pf. Let $x_1 \le x_2$ for $x_1, x_2 \in \mathbb{R}$.
Since g is nondecreasing $g(x_1) \le g(x_2)$.
Since f is nonincreasing $f(x_1) \ge f(x_2)$, which
multiplied by -1 gives $-f(x_1) \le -f(x_2)$.
Adding inequalities gives $g(x_1) - f(x_1) \le g(x_2) - f(x_2)$
Hence $h(x_1) \le h(x_2)$ //

(b) (10 pts) Let $f : \mathbb{R} \to (2, \infty)$ and $g : (2, \infty) \to (0, \infty)$ be functions.
Define $h : \mathbb{R} \to (0, \infty)$ by $h(x) = \cancel{3g(f(x)-3)}$. CHANGE: $3g(f(x-4))$
Prove that if f is surjective and g is surjective, then h is surjective.

pf. Let $y \in (0, \infty)$. Then $y/3 \in (0, \infty)$ (That is $y > 0 \Rightarrow \frac{y}{3} > 0$)
Since g is onto, $\exists u \in (2, \infty)$ such that
$g(u) = y/3$ (so $3g(u) = y$).
Since f is onto, $\exists v \in \mathbb{R}$ such that
$f(v) = u$. (so $3g(f(v)) = y$).
Finally since, $v \in \mathbb{R}$, $x = v + 4$ is in $\mathbb{R}$ (so $x - 4 = v$)
Hence $3g(f(x-4)) = y$ and $x \in \mathbb{R}$.
Therefore $h(x) = y$ for some $x \in \mathbb{R}$ and h is onto //