Math 300 Winter 2011 Final

Name: ____

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- USE YOUR TIME WISELY! START EACH PROBLEM BY OUTLINING THE PROOFS/IDEAS ON THE PAGE. SPEND NO MORE THAN 20 MINUTES ON EACH PAGE.
- Put your answers on the exam, there are 5 pages (you may use your own paper for scratch work). If you need more space for your answer use the back of the preceding page and indicate that you have done so.
- If you don't know how to prove something, then you should still let me know everything you know about the problem (that is, give definitions and facts you know about the problem and discuss different techniques to prove the theorem). Also, if you are stuck you should try some examples. Don't leave any question blank, give me the opportunity to give you credit for what you know!

GOOD LUCK!

1. (a) Give the negation, contrapositive, and converse of the following statement avoiding the word 'not' in your final answers. Then determine which of the statements are true (no proof required):

ORIGINAL: "$\forall a, n \in \mathbb{N}$, if $ax \equiv 2 \pmod{n}$ for some $x \in \mathbb{Z}$, then a is even or n is even."

(4 pts) NEGATION:

(3 pts) CONTRAPOSITIVE:

(3 pts) CONVERSE:

(3 pts) CIRCLE **ALL** THAT ARE **TRUE**:

ORIGINAL NEGATION CONTRAPOSITIVE CONVERSE

(b) (3 pts) Suppose that P and Q are TRUE statements and R and S are FALSE statements. Determine the truth values of the following statements.

i. $(P \text{ or } R)$ and $(Q \text{ and } S)$	TRUE	FALSE
ii. $P \Rightarrow \neg(R \text{ or } Q)$	TRUE	FALSE
iii. $R \Rightarrow (P \text{ and } Q)$	TRUE	FALSE

(c) (6 pts) Give a counterexample (explain why it is a counterexample)

i. Every monotone function from $\mathbb{R}$ to $\mathbb{R}$ is surjective.

ii. Every function from the set $A = \{1, 2, 3\}$ to $B = \{1, 2, 3\}$ is a bijection.

2. (a) (8 pts) Using the definitions of even and odd, give a **proof by contradiction** for the statement: For all $x, y \in \mathbb{Z}$, if $x^2 + y + 5$ is odd, then x is odd or y is even.

(b) (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$ and define $g : \mathbb{R} - \{-1\} \to \mathbb{R}$ by $g(x) = 3f\left(\frac{x}{x+1}\right) - 1$.

Prove that if f is one-to-one, then g is one-to-one.

3. (18 pts)
Suppose $f : A \to B$ and $g : B \to C$ are functions and define $h : A \to C$ by $h(x) = g(f(x))$ for all $x \in A$. Assume all the sets A, B and C are the set of real numbers $\mathbb{R}$ (labeled differently just for easy reference in your proof).

Two of the statements below are TRUE and two of the statements are FALSE. Correctly circle which are TRUE and which are FALSE.

(a) If f and g are decreasing, then h is decreasing.	TRUE	FALSE
(b) If f and g are surjective, then h is surjective.	TRUE	FALSE
(c) If g is bounded, then h is bounded.	TRUE	FALSE
(d) If f is bounded, then h is bounded.	TRUE	FALSE

- Provide counterexamples for both the statements that you said were false.

- Give a proof for ONE of the statements you said was true, you can pick whichever you want. (If you attempt to prove both, clearly indicated which one you want me to grade. Otherwise, I will average the grades on your two proofs. If you give completely correct proofs for both true statements, I will award 2 bonus points.)

4. (8 pts) Your poorly designed robot has two walking functions.
It can either take large steps of exactly 33 inches forward or back, or it can take short steps of exactly 27 inches forward or back. Thus, the only distances that the robot can travel exactly are the distances that can be expressed as $33x + 27y$ for some integers x and y.

   (a) Using appropriate reference to a theorem from class, describe ALL the distances that the robot can exactly travel.

   (b) Find one solution, $x, y \in \mathbb{Z}$, to the equation $33x + 27y = 9$.
   (Hint: We have a systematic algorithm to do this.)

5. (12 pts) Let $k \in \mathbb{Z}$ with $k \geq 0$. Using induction on n, prove that for all integers $n \geq 0$,

$$\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1}.$$

6. (a) (8 pts) Prove that for all $n \in \mathbb{N}$, if 6 divides n, then 9 divides $3^n + 2^n - 1$.

(b) (4 pts) Find a counterexample to the statement:
For all $n \in \mathbb{N}$, $x^2 \equiv 1 \pmod{n}$ if and only if $x \equiv 1 \pmod{n}$ or $x \equiv -1 \pmod{n}$.
(Hint: You may have to try a few different values for n before you come up with a counterexample, it will help to peek at the next part).

(c) (10 pts) Let p be a prime number.
Prove that $x^2 \equiv 1 \pmod{p}$ **if and only if** $x \equiv 1 \pmod{p}$ or $x \equiv -1 \pmod{p}$.
(Hint: It might help to rewrite it as a divisibility problem for one of the directions.)