1. (a) Give the negation, contrapositive, and converse of the following statement avoiding the word 'not' in your final answers. Then determine which statement is true (no proof required): ORIGINAL: "For all $a, b \in \mathbb{Z}$, if $a^2 + b^2$ is odd, then a is even or b is even."

(4 pts) NEGATION:

$\exists_{a,b \in \mathbb{Z}}$ s.t. a^2+b^2 is ODD AND a is ODD AND b is ODD.

(3 pts) CONTRAPOSITIVE:

$\forall_{a,b \in \mathbb{Z}}$, if a is ODD AND b is ODD, then a^2+b^2 is EVEN.

(3 pts) CONVERSE:

$\forall_{a,b \in \mathbb{Z}}$, if a is EVEN OR b is EVEN, then a^2+b^2 is ODD.

(3 pts) CIRCLE **ALL** THAT ARE **TRUE**:

ORIGINAL NEGATION CONTRAPOSITIVE CONVERSE

(b) (3 pts) Find a counterexample to the following statement:
For all $x, y, z \in \mathbb{N}$, if $x + y + z = 10$, then $xyz > 10$.

The only counterexample (up to ordering) is

$\boxed{x = 1,\ y = 1,\ z = 8}$

$1+1+8=10$ ✓

$1 \cdot 1 \cdot 8 \leq 10$ ✓

(c) (6 pts) Fill in the truth table below with the appropriate truth values in all cases.

P	Q	$\neg P$	$\neg Q$	$P \wedge \neg Q$	$\neg Q \Rightarrow (P \wedge \neg Q)$	$\neg P \vee Q$	$(\neg P \vee Q) \Rightarrow Q$
T	T	F	F	F	T	T	T
T	F	F	T	T	T	F	T
F	T	T	F	F	T	T	T
F	F	T	T	F	F	T	F

(2 pts) How is the statement $\neg Q \Rightarrow (P \wedge \neg Q)$ related to the statement $(\neg P \vee Q) \Rightarrow Q$? Clearly circle **all** that apply.

i. They are converses of each other.
ii. They are contrapositives of each other.
iii. They are the negation of each other.
iv. They are logically equivalent.

2. (a) (13 pts) Let A, B, C, and D be sets.
By giving a formal subset proof, prove that $[A-(B\cap C)]\cap D \subseteq (A-C)\cup B^c$.

pf Assume $x \in [A-(B\cap C)]\cap D$.

By def'n of intersect, $x \in A-(B\cap C)$ AND $x \in D$.

By def'n of difference, $x \in A$ AND $x \in (B\cap C)^c$ AND $x \in D$.

By deMorgan's law, $x \in A$ AND $x \in B^c \cup C^c$ AND $x \in D$.

By def'n of union, $x \in A$ AND ($x \in B^c$ OR $x \in C^c$) AND $x \in D$.

Thus one of two cases is true (by def'n of OR)

CASE 1: $x \in A$ AND $x \in B^c$ AND $x \in D$. In particular, $x \in B^c$, so $x \in (A-C)$ OR $x \in B^c$. Hence $x \in (A-C)\cup B^c$.

CASE 2: $x \in A$ AND $x \in C^c$ AND $x \in D$. In particular $x \in A$ AND $x \in C^c$, so $x \in A-C$. Thus, $x \in A-C$ OR $x \in B^c$.

Hence in all cases, $x \in (A-C)\cup B^c$ //

(b) (10 pts) Prove if $f(x)$ and $g(x)$ are bounded, then $(f(x))^2 + 3g(x)$ is bounded.

pf Assume f and g are bounded. By def'n, there exists $M_1, M_2 \in \mathbb{R}$ such that $|f(x)| \le M_1$ and $|g(x)| \le M_2$ $\forall x \in \mathbb{R}$.

multiplying the first inequality by itself (and noting all both sides are positive) gives $|f(x)|^2 \le M_1^2$.

multiplying the second inequality by 3 gives $3|g(x)| \le 3M_2$.

By the triangle inequality,

by def'n of absolute value and result from lecture.

$$|f(x)^2 + 3g(x)| \le |f(x)^2| + |3g(x)| = |f(x)|^2 + 3|g(x)|$$

And adding the inequalities from above, $|f(x)|^2 + 3|g(x)| \le M_1^2 + 3M_2$.

Hence, by the transitive property, $|f(x)^2 + 3g(x)| \le M_3 = M_1^2 + 3M_2$.

Ergo, $f(x)^2 + 3g(x)$ is bounded //

3. (a) (4 pts) Let $f(x) = \sin(x)$.
Find two specific sets A and B such that $f(A \cap B)$ is not equal to $f(A) \cap f(B)$.
(For your sets, give the sets $f(A \cap B)$ and $f(A) \cap f(B)$ as well).

$A = [0, \pi/2]$
$B = [\pi/2, \pi]$
$f(A \cap B) = f(\{\pi/2\}) = \{1\}$
$f(A) \cap f(B) = [0,1] \cap [0,1] = [0,1]$

MANY OTHER CORRECT SOL'NS CAN BE FOUND. HERE'S ANOTHER
$A = \{0\}$ $B = \{\pi\}$
$f(A \cap B) = f(\emptyset) = \emptyset$
$f(A) \cap f(B) = \{0\} \cap \{0\} = \{0\}$

(b) (14 pts) Using induction on n, prove that for all $n \in \mathbb{N}$ with $n \geq 3$ we have

$$\prod_{i=3}^{n} \left(1 - \frac{4}{i^2}\right) = \frac{(n+1)(n+2)}{6(n-1)n}.$$

$\boxed{\text{BASE STEP}}$ For $n=3$, $\prod_{i=3}^{3} \left(1 - \frac{4}{i^2}\right) = 1 - \frac{4}{9} = \frac{5}{9}$

and $\frac{(3+1)(3+2)}{6(3-1)\cdot 3} = \frac{\cancel{4}\cdot 5}{\cancel{6}\cdot\cancel{2}\cdot 3} = \frac{5}{9}$ ✓

$\boxed{\text{IND. STEP}}$ Assume $\prod_{i=3}^{k} \left(1 - \frac{4}{i^2}\right) = \frac{(k+1)(k+2)}{6(k-1)k}$ for some $k \in \mathbb{N}$, $k \geq 3$

Then
$$\prod_{i=3}^{k+1} \left(1 - \frac{4}{i^2}\right) = \left(1 - \frac{4}{(k+1)^2}\right) \prod_{i=3}^{k} \left(1 - \frac{4}{i^2}\right)$$ (pulling out last term)

$$= \left(1 - \frac{4}{(k+1)^2}\right) \frac{(k+1)(k+2)}{6(k-1)k}$$ $\boxed{\text{by ind. hyp.}}$

$$= \frac{((k+1)^2 - 4)}{(k+1)^{\cancel{2}}} \frac{\cancel{(k+1)}(k+2)}{6(k-1)k}$$ (common denominator)

$$= \frac{(k^2+2k-3)(k+2)}{6(k+1)(k-1)k}$$ (expanding)

$$= \frac{(k+3)\cancel{(k-1)}(k+2)}{6(k+1)\cancel{(k-1)}k}$$ (factoring)

$$= \frac{(k+1+1)(k+1+2)}{6(k+1-1)(k+1)}$$ (rewriting)

Thus, $\prod_{i=3}^{n} \left(1 - \frac{4}{i^2}\right) = \frac{(n+1)(n+2)}{6(n-1)n}$. $\forall n \in \mathbb{N}$, $n \geq 3$ //

$\boxed{\text{ASIDE}}$ THUS, $\prod_{i=3}^{\infty} \left(1 - \frac{4}{i^2}\right) = \lim_{n \to \infty} \frac{(n+1)(n+2)}{6(n-1)n} = \frac{1}{6}$

4. (15 pts) Let x, y, b and m be integers such that $y = mx + b$ such that **b is even**. Clearly giving a well structure proof and using the precise definitions of even and odd, prove m is odd and x is odd **if and only if** y is odd.
(Use an indirect method for one direction).

pf| Let b be even so $b = 2k$ for some $k \in \mathbb{Z}$.
We must show both directions.

(1) (m odd and x odd $\Rightarrow$ y odd)

(1) Assume m is odd and x is odd.

Then $m = 2l+1$ and $x = 2p+1$ for some $l, p \in \mathbb{Z}$.
By substitution into the given fact,

$$y = mx + b = (2l+1)(2p+1) + 2k$$
$$= 4lp + 2l + 2p + 1 + 2k$$
$$= 2(2lp + l + p + k) + 1.$$

Hence, y is odd.

(2) (y odd $\Rightarrow$ m odd and x odd)

(2) We prove the contrapositive for this direction.

Assume m is even OR x is even.

CASE 1 m is even. Then $m = 2q$ for some $q \in \mathbb{Z}$.
By substitution, $y = mx + b = 2qx + 2k = 2(qx + k)$.

CASE 2 x is even, Then $x = 2r$ for some $r \in \mathbb{Z}$.
By substitution, $y = mx + b = m(2r) + 2k = 2(mr + k)$.

Hence, in all cases, y is even.//