Math 300 Summer 2010 Final Exam

Name: ______________________________

1	31	
2	23	
3	21	
4	25	
Total	100	

- USE YOUR TIME WISELY! START EACH PROBLEM BY OUTLINING THE PROOFS/IDEAS ON THE PAGE. THE LAST TWO PAGES ARE PROOFS. SPEND NO MORE THAN 10-20 MINUTES ON THE FIRST TWO PAGES.
- Put your answers on the exam, there are 4 pages (you may use your own paper for scratch work). If you need more space for your answer use the back of the preceding page and indicate that you have done so.
- You can pick up your exams anytime during fall quarter (check my website for office hours). Or you can email me in the fall and I will leave your exam in an envelope outside my office.
- If you don't know how to prove something, then you should still let me know everything you know about the problem (that is, give definitions and facts you know about the problem and discuss different techniques to prove the theorem). Also, if you are stuck you should try some examples. Don't leave any question blank, give me the opportunity to give you credit for what you know!

GOOD LUCK!

1. (a) (12 pts) Let $f : A \to B$ and $g : B \to C$ be functions, where $A, B, C \subseteq \mathbb{R}$. For the definitions below, identify the definition (tell me the name of what is being defined) and give the negation of the statement

   i. $\forall x_1, x_2 \in A$, if $x_1 < x_2$, then $f(x_1) > f(x_2)$. NAME OF DEF'N:

      NEGATION:

   ii. $\exists M \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$, $|f(x)| \leq M$. NAME OF DEF'N:

      NEGATION:

   (b) (4 pts) Give a specific counterexample to the statement:
   Every injective function from $\mathbb{R}$ to $\mathbb{R}$ is not bounded.

   (c) Consider the statement:
   For all $n, a, b \in \mathbb{N}$, if $ab \equiv 0 \pmod{n}$, then $a \equiv 0 \pmod{n}$ or $b \equiv 0 \pmod{n}$.

   i. (4 pts) Give a specific counterexample to the statement.

   ii. (3 pts) Give a condition on n that makes the statement true.

   (d) (8 pts) Use congruence arithmetic to simplify and solve for an integer x such that $0 \leq x < 7$ and $6^{411}x + 8^{911} + 2 \equiv 23^6 \pmod{7}$. (You must show your work to get credit).

2. (a) (12 pts) Let A, B, and C be sets.
Using a formal, and properly structure, subset proof with proper reference to definitions, logic and de'Morgan's law, prove $A \cap (B - (A \cap C)) \subseteq B \cap (A - C)$.

(b) Let $f : A \to B$ and $g : B \to C$ be functions.
Define $h : A \to C$ by $h(x) = g(f(x))$ for all $x \in A$.

i. (6 pts) Give a specific counterexample (give me your functions and sets) to the statement: If h is surjective, then f is surjective.

ii. (5 pts) Consider the following theorem. *Theorem:* If h is injective, then f is injective. Now is your chance to be a proof grader. Of the three "proofs" below, only ONE is correct. Which is the correct proof? And why?

('Proof' 1) Assume $h(x_1) = h(x_2)$. By definition of h, $g(f(x_1)) = g(f(x_2))$. Since h is injective, $x_1 = x_2$, so $f(x_1) = f(x_2)$. Since we have $f(x_1) = f(x_2)$ and $x_1 = x_2$, f is injective.
('Proof' 2) Assume $x_1 = x_2$ for $x_1, x_2 \in A$. Since f and g are well-defined, $f(x_1) = f(x_2)$ and $g(f(x_1)) = g(f(x_2))$. Thus, $h(x_1) = h(x_2)$. Since h is injective, we have $x_1 = x_2$, so f is injective.
('Proof' 3) Assume $f(x_1) = f(x_2)$. Since g is well-defined, $g(f(x_1)) = g(f(x_2))$. Thus, $h(x_1) = h(x_2)$. Since h is injective, $x_1 = x_2$. Hence, f is injective.

ANSWER AND EXPLANATION:

3. (a) (10 pts) By using a precisely worded induction proof, prove $3^n > 2^{n+1}$ for all integers $n \geq 2$.

(b) (10 pts) $\forall a, b, c, d \in \mathbb{N}$, prove if $\gcd(a + b, c) = 2d$, $\gcd(a, b) = 28$, and $14|c$, then $7|d$.

4. (a) Let a, b and c be integers.

   i. (6 pts) Using the definition of even and odd, prove if c^3 is even, then c is even. (Hint: Prove the contrapositive.)

   ii. (10 pts) Using a proof by contradiction, prove if $(2a-1)^2 + (2b-1)^2 = c^3$, then a is odd or b is odd.

   (b) (9 pts) Prove if p is a prime number and $p > 4$, then $p^2 - 1 \equiv 0 \pmod{12}$. (Hint: Think about the possible remainders when p is divided by 12.)