1. (a) (5 pts) Let P and Q be statements. There are four possible combinations of truth values for P and Q, as listed below. Determine which combinations make the following statement true (circle the cases that make this statement true): $(P \vee Q) \Rightarrow \neg(P \wedge Q)$.

Justify your work, by constructing an appropriate truth table for this statement.

	P	Q	$P \vee Q$	$P \wedge Q$	$\neg(P \wedge Q)$	$(P \vee Q) \Rightarrow \neg(P \wedge Q)$
I.	T	T	T	T	F	F
II.	T	F	T	F	T	T
III.	F	T	T	F	T	T
IV.	F	F	F	F	T	T

(b) Give the negation, contrapositive, and converse of the following statement avoiding the word 'not' in your final answers where possible. Then determine which statement(s) are true (no proof required):

ORIGINAL: "For all $x, y \in \mathbb{R}$, if $xy < 0$, then $x > 0$ or $y > 0$."

(4 pts) NEGATION:

$\boxed{\text{There exists } x, y \in \mathbb{R} \text{ such that } xy < 0 \text{ AND } x \leq 0 \text{ AND } y \leq 0.}$

(3 pts) CONTRAPOSITIVE:

$\boxed{\text{For all } x, y \in \mathbb{R}, \text{ if } x \leq 0 \text{ AND } y \leq 0, \text{ then } xy \geq 0.}$

(3 pts) CONVERSE:

$\boxed{\text{For all } x, y \in \mathbb{R}, \text{ if } x > 0 \text{ OR } y > 0, \text{ then } xy < 0.}$

ASIDE: counterexample $x=2$, $y=3$!

(3 pts) CIRCLE **ALL** THAT ARE **TRUE**:

ORIGINAL   NEGATION   CONTRAPOSITIVE   CONVERSE

(c) (4 pts) Find a counterexample to the following statement (show your work verifying that it is a counterexample):

If a and b are integers and $a + b = 3$, then $\dfrac{(a+1)(b+2a)}{2}$ is an integer.

NEED
① $a, b \in \mathbb{Z}$
② $a+b=3$
③ $\frac{(a+1)(b+2a)}{2}$ is not an integer

$\boxed{\text{ANY EXAMPLE WITH } a \text{ EVEN and } b = 3-a \text{ WILL BE SUCH A COUNTEREXAMPLE}}$

Ex) $a=2,\ b=1$

$a+b=3$ ✓

$\frac{(a+1)(b+2a)}{2} = \frac{(3)(5)}{2} = 7.5 \notin \mathbb{Z}$

2. (a) (6 pts) Let $A = \{-2, 3, 5\}$, $B = \{-1, 2, 5\}$ and $f : \mathbb{Z} \to \mathbb{Z}$ be the function $f(x) = x^2$. Circle all the statements below for which these particular sets A and B and this particular function $f(x)$ provide a counterexample (justify your answer by showing the resultant sets):

   i. For all sets A and B, $f(A) \cap f(B) \subseteq f(A \cap B)$.

   ii. For all sets A and B, $f(A \cap B) \subseteq f(A) \cap f(B)$.

   iii. For all sets A and B, $f(A) - f(B) \subseteq f(A - B)$.

   iv. For all sets A and B, $f(A - B) \subseteq f(A) - f(B)$.

$f(A) = \{4, 9, 25\}$, $f(B) = \{1, 4, 25\} \Rightarrow f(A) \cap f(B) = \{4, 25\}$

$f(A) - f(B) = \{9\}$

$A \cap B = \{5\} \Rightarrow f(A \cap B) = \{25\}$

$A - B = \{-2, 3\} \Rightarrow f(A-B) = \{4, 9\}$

For this example, $f(A \cap B) \subseteq f(A) \cap f(B)$, BUT $f(A) \cap f(B) \not\subseteq f(A \cap B)$

also $f(A) - f(B) \subseteq f(A - B)$, BUT $f(A-B) \not\subseteq f(A) - f(B)$

(b) (12 pts) Let A, B, and C be sets.
Give a **formal subset proof** using only definitions and facts from your fact sheet to prove that

$$(B \cap C) \cup [C - (A \cup B)] \subseteq (A - B)^c.$$

pf/ Let $x \in (B \cap C) \cup [C - (A \cup B)]$.

By definition of union, $x \in B \cap C$ OR $x \in C - (A \cup B)$.

Hence, one of the following cases must be true:

$\boxed{\text{CASE I}}$ If $x \in B \cap C$, then, in particular, $x \in B$.

Since $x \in B$ is TRUE, ($x \in A^c$ OR $x \in B$) IS TRUE.

Hence, $x \in A^c \cup B = (A \cap B^c)^c$ by de Morgan's Law.

$\boxed{\text{CASE II}}$ If $x \in C - (A \cup B)$, then, by def'n of set difference,

$x \in C$ AND $x \in (A \cup B)^c$; By de Morgan's Law $(A \cup B)^c = A^c \cap B^c$.

Thus, $x \in C$ AND $x \in A^c$ AND $x \in B^c$.

In particular, $x \in A^c$ IS TRUE, so ($x \in A^c$ OR $x \in B$) IS TRUE.

Therefore $x \in A^c \cup B = (A \cap B^c)^c$ as above.

In all cases, $x \in (A \cap B^c)^c = (A - B)^c$ //

3. (a) (14 pts) Using **induction** on n, prove that for all $n \in \mathbb{N}$, we have $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$.

pf $\boxed{\text{BASE STEP}}$ For $n=1$, $\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1(1+1)} = \frac{1}{2}$ and $\frac{1}{1+1} = \frac{1}{2}$.

Thus, $\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1+1}$.

$\boxed{\text{IND. STEP}}$ Assume $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$ for some $k \in \mathbb{N}$.

Therefore,

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{1}{(k+1)(k+2)} + \sum_{i=1}^{k} \frac{1}{i(i+1)} \quad \left\{ \text{pulling out the last term} \right.$$

$$= \frac{1}{(k+1)(k+2)} + \frac{k}{(k+1)} \quad \left\{ \text{by the inductive hypothesis} \right.$$

$$= \frac{1+k(k+2)}{(k+1)(k+2)} \quad \left\{ \text{common denominator} \right.$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)} \quad \text{simplifying}$$

$$= \frac{(k+1)^{\cancel{2}}}{(\cancel{k+1})(k+2)} \quad \text{factoring}$$

$$= \frac{k+1}{(k+1)+1}$$

By mathematical induction, $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ $\forall n \in \mathbb{N}$ //

(b) (10 pts) Let a, b, and c be integers. Using a **proof by contradiction** and the definitions of even and odd, prove that if $a^2 + b^2 = 4c$, then a is even or b is even.

pf Let a, b, and $c \in \mathbb{Z}$ such that $a^2+b^2=4c$.

Assume a is ODD AND b is ODD.

By def'n of odd, $a = 2k+1$ and $b = 2l+1$ for some $k, l \in \mathbb{Z}$.

By substitution, $(2k+1)^2 + (2l+1)^2 = 4c$.

Expanding gives, $4k^2+4k+1+4l^2+4l+1 = 4c$

$$2 = 4c - 4k^2 - 4k - 4l^2 - 4l$$

$$2 = 4(c-k^2-k-l^2-l)$$

$$\frac{1}{2} = c-k^2-k-l^2-l. \quad \rightarrow\leftarrow$$

there is never an integer sum equal to 1/2

Therefore a is ODD AND b is ODD is FALSE, so

a is even or b is even is TRUE //

$\mathbb{Z}$

4. (16 pts) Give a carefully organized and complete proof that uses the formal definitions of even and odd to prove the following statement:
For all $x, y \in \cancel{\mathbb{R}}$, x is even and y is odd **if and only if** $x + y$ is odd and $x + 2y$ is even.

We must show ① x even, y odd $\Rightarrow$ $x+y$ odd, $x+2y$ even
② $x+y$ odd, $x+2y$ even $\Rightarrow$ x even, y odd.

pf ① Assume x is even and y is odd.
By def'n, $x = 2k$ and $y = 2l+1$ for some $k, l \in \mathbb{Z}$.
By substitution, $x+y = 2k+2l+1 = 2(k+l)+1$ which is odd, and
$x+2y = 2k+2y = 2(k+y)$ which is even.

② Assume $x+y$ is odd and $x+2y$ is even.
By def'n, $x+y = 2m+1$ and $x+2y = 2n$ for some $m, n \in \mathbb{Z}$.
Since $x+2y = 2n$, rearranging gives $x = 2n - 2y = 2(n-y)$
which is even.
Subtracting the equations $y = (x+2y) - (x+y) = 2n - (2m+1)$
$= 2(n-m) - 1$
which is odd //