1. (a) (9 pts) Give a counterexample for each of the following statements. your counterexamples must be functions such that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$.

i. If f is unbounded and decreasing, then f is surjective.

$\boxed{f(x) = -e^x}$ or $f(x) = e^{-x}$ or $f(x) = \begin{cases} -x, & x<0 \\ -x-1, & x \geq 0 \end{cases}$

ii. If f is surjective and g is surjective, then fg is surjective.

JUST LIKE HW 4.24 → $\boxed{f(x) = x, g(x) = x}$ $f(x)g(x) = x^2$

or many others

iii. If f is injective and g is injective, then $f + g$ is injective.

$\boxed{\begin{array}{l} f(x) = x^3 \\ g(x) = -x \end{array}}$ $f(x)+g(x) = x^3 - x$

or $\boxed{\begin{array}{l} f(x) = x \\ g(x) = -x \end{array}}$ $(f+g)(x) = 0$

(b) (5 pts) For $n = 14$ and $k = 12$, verify through calculation that $\dfrac{n}{\gcd(n,k)}$ divides $\binom{n}{k}$. (There is no proof to give here, I just want to see that you verify the result for the particular values $n = 14$ and $k = 12$.)

$\gcd(14,12) = 2$

$\dfrac{n}{\gcd(14,12)} = \dfrac{14}{2} = 7$

$\binom{n}{k} = \binom{14}{12} = \dfrac{14!}{12!\,2!} = \dfrac{14 \cdot 13}{2 \cdot 1} = 7 \cdot 13 = 91$

So $\dfrac{14}{\gcd(14,12)}$ divides $\binom{14}{12}$ because $91 = 7(13)$.

(c) (5 pts) Using the binomial theorem and Pascal's triangle to help you expand, prove that for all $x, y \in \mathbb{Z}$, if $\gcd(x,y) = 2$, then the number $x^4 - (x-y)^4 + y^4$ is divisible by 32.

Should say what hypothesis means

→ Since $\gcd(x,y) = 2$, $2 \mid x$ and $2 \mid y$. By def'n, $x = 2k$ and $y = 2\ell$ for some $k, \ell \in \mathbb{Z}$. By the binomial theorem

Expand

$$\begin{aligned} x^4 - (x-y)^4 + y^4 &= \cancel{x^4} - (\cancel{x^4} - 4xy^3 + 6x^2y^2 - 4xy^3 + \cancel{y^4}) + \cancel{y^4} \\ &= 4(2k)^3 2\ell - 6(2k)^2(2\ell)^2 + 4(2k)(2\ell)^3 \\ &= 4 \cdot 2^4 k^3 \ell - 6 \cdot 2^4 k^2 \ell^2 + 4 \cdot 2^4 k \ell^3 \\ &= 2^5 (2k^3\ell - 3k^2\ell^2 + 2k\ell^3) \\ &= 32\, d \end{aligned}$$

for some integer $d = 2k^3\ell - 3k^2\ell^2 + 2k\ell^3 \in \mathbb{Z}$.

So 32 divides $x^4 - (x-y)^4 + y^4$.

2. (a) (8 pts) Consider $f : A \to B$, $g : B \to C$, and $h = g \circ f : A \to C$, where A, B and C are subsets of $\mathbb{R}$. Indicate which statements are true and which are false (no proof needed):

i. If h is bijective, then f is surjective. SEE HW 4.34c FOR COUNTEREXAMPLE — TRUE — FALSE

ii. If h is surjective, then g is surjective. SEE HW 4.34d FOR PROOF — TRUE — FALSE

iii. If f is decreasing and g is decreasing, then h is decreasing. SEE HW FOR PROOF inc. — TRUE — FALSE

iv. If f is increasing and g is increasing, then h is increasing. THINK ABOUT SIMILAR HW PROBLEM — TRUE — FALSE

(b) (8 pts) Let $f : A \to B$, $g : B \to C$, and $h = g \circ f : A \to C$ be functions. Consider the following theorem. *Theorem:* If h is injective, then f is injective.

Now is your chance to be a proof grader. Of the four "proofs" below, only ONE is correct. **Tell me which proof is correct and for each of the other proofs give a specific reason that the proof is incorrect.**

('Proof' 1) Let $x_1, x_2 \in A$ such that $h(x_1) = h(x_2)$. Since h is injective, $x_1 = x_2$. Since f is well-defined, $f(x_1) = f(x_2)$. Thus, we have $f(x_1) = f(x_2)$ and $x_1 = x_2$. $\square$
('Proof' 2) Let $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$. Since g is well-defined, $g(f(x_1)) = g(f(x_2))$. By definition, $h(x_1) = h(x_2)$. Since h is injective, $x_1 = x_2$. $\square$
('Proof' 3) Let $x_1, x_2 \in A$ such that $x_1 = x_2$. Since f is well-defined, $f(x_1) = f(x_2)$. Since g is well-defined, $g(f(x_1)) = g(f(x_2))$. By definition, $h(x_1) = h(x_2)$. Since h is injective, $x_1 = x_2$. $\square$
('Proof' 4) Let $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$. Since g is well-defined, $g(f(x_1)) = g(f(x_2))$. Since h is injective, $f(x_1) = f(x_2)$. Since $f(x_1) = f(x_2)$, we have $x_1 = x_2$. $\square$

ANSWER AND EXPLANATION:

PROOF 1 IS WRONG, IT DOES NOT START WITH $f(x_1) = f(x_2)$ TO SHOW $x_1 = x_2$.

PROOF 2 IS CORRECT

PROOF 3 IS WRONG, IT PROVES NOTHING, IT STARTS WITH $x_1 = x_2$ and ends will $x_1 = x_2$.

PROOF 4 IS WRONG, IT ASSUMES g IS INJECTIVE WHEN IT GOES FROM $g(f(x_1)) = g(f(x_2))$ to $f(x_1) = f(x_2)$.

IF YOU HAVE BEEN STUDYING AND DOING THE HOMEWORK, THEN YOU SHOULD KNOW THAT THE PROOF MUST START WITH $f(x_1) = f(x_2)$

CHECK YOUR TIME! LEAVE 20 MINUTES FOR THE LAST PAGE!

3. (a) (13 pts) Consider the sequence defined by $a_1 = 2$, $a_2 = 12$, and $a_n = a_{n-1}a_{n-2} + 20a_{n-2}$ for $n \geq 3$. Using the precise phrasing for strong induction, prove that 2^n divides a_n for all $n \in \mathbb{N}$.

pf| BASE STEP For $n=1$, $2^n = 2^1$ divides $a_1 = 2$ because $a_1 = 2 = 2(1) = 2^1(1)$.

For $n=2$, $2^n = 2^2 = 4$ divides $a_2 = 12$ because $a_2 = 12 = 4(3) = 2^2(3)$.

IND. STEP. Assume 2^i divides a_i for $i = 1, 2, \ldots, k$ for some $k \in \mathbb{N}$, $k \geq 2$.

By the defining recurrence, $a_{k+1} = a_k a_{k-1} + 20 a_{k-1}$.

By the ind. hyp., $2^k \mid a_k$ and $2^{k-1} \mid a_{k-1}$, so $\exists p, q \in \mathbb{Z}$ such that $a_k = 2^k p$ and $a_{k-1} = 2^{k-1} q$

By substitution, $a_{k+1} = 2^k p\, 2^{k-1} q + 20\, 2^{k-1} q = 2^{2k-1} pq + 2^{k+1} 5q$
$= 2^{k+1}(2^{k-2} pq + 5q)$ and $2^{k-2} pq + 5q$ is an integer because $k-2 \geq 0$

Hence, 2^{k+1} divides a_{k+1} //

VERY MUCH LIKE HW 3.56

(b) (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Prove if f is increasing, then f is injective.

pf| Assume f is increasing.

Let $\boxed{f(x_1) = f(x_2)}$ with $x_1, x_2 \in \mathbb{R}$.

We must show that $x_1 = x_2$.

If $x_1 \neq x_2$, then either $x_1 < x_2$ or $x_1 > x_2$.

Since f is increasing, if $x_1 < x_2$, then $f(x_1) < f(x_2)$.
and if $x_1 > x_2$, then $f(x_1) > f(x_2)$.

Hence, if $x_1 < x_2$ or $x_1 > x_2$, then $f(x_1) \neq f(x_2)$.

Since $f(x_1) = f(x_2)$, this can't be the case (that is $x_1 < x_2$ IS FALSE and $x_1 > x_2$ IS FALSE)

Thus, $\boxed{x_1 = x_2}$ //

WE DID THIS IN LECTURE

Similar in approach to 4.3

No matter what you should have written the first and last step and the def'n of increasing

4. (a) (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a function for which there exists positive real numbers a and b such that

$$a(f(x) - f(y)) - b > x - y \text{ for all } x, y \in \mathbb{R}.$$

Prove that f is increasing on $\mathbb{R}$.

pf Assume $x_1, x_2 \in \mathbb{R}$ with $\boxed{x_1 < x_2}$. Letting $x = x_2$ and $y = x_1$, in the given relationship we have

$a(f(x_2) - f(x_1)) - b > x_2 - x_1$. Since $x_1 < x_2$, $x_2 - x_1 > 0$.

Thus, $a(f(x_2) - f(x_1)) - b > 0$. Simplifying gives

$f(x_2) - f(x_1) > \frac{b}{a}$ (allowed because $a > 0$).

Since $a, b > 0$, $\frac{b}{a} > 0$, so $f(x_2) - f(x_1) > 0$.

Thus, $f(x_2) > f(x_1)$ so $\boxed{f(x_1) < f(x_2)}$ //

Should have written first and last step

Similar in approach to 3.41 and 4.26

(b) (12 pts) Let $f : A \to B$, $g : B \to C$, and $h = g \circ f : A \to C$ be functions. Prove that if g is bijective and h is bijective, then f is surjective. (Be very clear about the order, justifications and sets you are referring to in your proof).

pf Let $\boxed{b \in B}$.

Since g is a function defined from B to C, $g(b) = c$ for some $c \in C$.

Since h is surjective, there exists an $a \in A$ such that $h(a) = c$.

By definition of h, $g(f(a)) = c$.

Thus, $g(b) = c = g(f(a))$, so $g(b) = g(f(a))$.

Since g is injective, $b = f(a)$.

Hence, $\boxed{f(a) = b \text{ for some } a \in A}$ //

We did an identical problem during the review

This appeared in almost identical form on a previous midterm

like 4.34 of HW

Should have known start and end