Math 300 Winter 2011 Exam 1

Name: Solutions

Student ID Number: ______________________

PAGE 1	26	
PAGE 2	14	
PAGE 3	24	
PAGE 4	16	
Total	80	

- USE YOUR TIME WISELY! START EACH PROBLEM BY OUTLINING THE PROOFS/IDEAS ON THE PAGE. THEN SPEND NO MORE THAN 10 MINUTES PER PAGE FILLING IN THE DETAILS (AND THEN MOVE ON TO THE NEXT PAGE).
- I will primarily be giving points for structure and format of proofs. Keep your proofs minimal, but make sure to justify each step.
- Put your answers on the exam, there are 4 pages (you may use your own paper for scratch work, but everything you want grade must be on the exam). If you need more space for your answer use the back of the preceding page and indicate that you have done so.
- If you don't know how to prove something, then you should still let me know everything you know about the problem (that is, give definitions and facts you know about the problem and discuss different techniques to prove the theorem). Also, if you are stuck you should try some examples. Don't leave any question blank, show me what you know!

GOOD LUCK!

1. (a) (12 pts) For each statement below, give the negation AND tell me which statement is true.

i. There exists $a \in \mathbb{N}$ such that, for all $b \in \mathbb{N}$, if ab is even, then $a + b$ is odd. ← TRUE, CHOOSE a ODD

$\boxed{\forall a \in \mathbb{N}, \exists b \in \mathbb{N} \text{ s.t. } ab \text{ is even AND } a+b \text{ is even}}$ ← FALSE, FOR a ODD ITS NOT POSSIBLE TO FIND SUCH A b.

CIRCLE WHICH IS TRUE **TRUE:** ORIGINAL NEGATION

ii. For all $x, y \in \mathbb{R}$, if $x + y > 10$, then $x > 0$ or $y > 0$. ← TRUE, AT LEAST ONE MUST BE POSITIVE

$\boxed{\exists x, y \in \mathbb{R} \text{ st } x+y > 10 \text{ AND } x \leq 0 \text{ AND } y \leq 0.}$ ← FALSE, TWO NEGATIVES CAN'T SUM TO LARGER THAN 10.

CIRCLE WHICH IS TRUE **TRUE:** ORIGINAL NEGATION

2. (14 pts) Answer the following questions about the statement: For all $n \in \mathbb{N}$, for all $q \in \mathbb{R}$, $nq \leq q^n$.

(a) Assume we tried to give a proof by induction on n.

i. (4 pts) Is the base step true? (Explain).

For $n = 1$ and for any q, $nq = q \leq q = q^n$.
Thus, the base step is $\boxed{\text{TRUE}}$.

ii. (6 pts) Properly state the regular inductive hypothesis and the strong inductive hypothesis for this statement.

REGULAR INDUCTIVE HYPOTHESIS:

$\boxed{\text{Assume } kq \leq q^k \text{ for some } k \in \mathbb{N}.}$

STRONG INDUCTIVE HYPOTHESIS:

$\boxed{\text{Assume } iq \leq q^i \text{ for } i = 1, 2, \ldots, k \text{ for some } k \in \mathbb{N}.}$

(b) (4 pts) The statement is false. Give a counterexample.

$\boxed{n = 2, q = 1}$

$\left.\begin{array}{l} nq = 2 \\ q^n = 1 \end{array}\right\}$ so $nq \not\leq q^n$

3. (14 points) Let A, B, and C be sets. Give a properly formatted subset proof, based on definitions and logic, that

$$(A^c \cup (B - C))^c \subseteq C \cup (A - B).$$

Let $x \in (A^c \cup (B-C))^c$.

By de Morgan's law, $x \in A \cap (B-C)^c$.

By def'n of set difference (and the proof from class) $B - C = B \cap C^c$. Thus, $x \in A$ AND $x \in (B \cap C^c)^c$

By de Morgan's law again, $x \in A$ AND $x \in B^c \cup C$.

Hence, $x \in A$ AND ($x \in B^c$ OR $x \in C$).

We consider the two cases:

1. $x \in A$ AND $x \in B^c$.
   By def'n of set difference, $x \in A - B$.
   Thus, $x \in C$ OR $x \in A - B$ is also true (by def'n of OR).
   So $x \in C \cup (A - B)$.

2. $x \in A$ AND $x \in C$
   In particular since $x \in C$, we have $x \in C$ or $x \in A - B$.
   Thus, $x \in C \cup (A - B)$.

In all cases, $x \in C \cup (A - B)$

Therefore $(A^c \cup (B - C))^c \subseteq C \cup (A - B)$ //

4. (9 pts) Give a carefully organized proof based only on known facts from the fact sheet that for all real numbers a and b, $|\sqrt{|a||b|}+a+b| \leq \frac{3}{2}(|a|+|b|)$.

pf] Let $a, b \in \mathbb{R}$.

By the triangle inequality (with $x = \sqrt{|a||b|}$ and $y = a+b$), we have

$$|\sqrt{|a||b|} + a + b| \leq |\sqrt{|a||b|}| + |a+b|.$$

By the triangle inequality again (with $x=a$ and $y=b$) we have

$$|\sqrt{|a||b|}| + |a+b| \leq |\sqrt{|a||b|}| + |a| + |b|.$$

Since the square root is a positive function $|\sqrt{|a||b|}| = \sqrt{|a||b|}$

By the AGM(c), $\sqrt{|a||b|} \leq \frac{|a|+|b|}{2}$. Putting this together with our previous inequalities (by the transitive property),

$$|\sqrt{|a||b|} + a + b| \leq \frac{|a|+|b|}{2} + |a| + |b| = \frac{3}{2}(|a|+|b|) //$$

5. (15 pts) Define a sequence by $a_1 = -1$ and $a_{n+1} = a_n + 2n - 1$ for all $n \in \mathbb{N}$. Using a formal proof by induction, prove that $a_n = n^2 - 2n$ for all $n \in \mathbb{N}$.

pf] BASE STEP: For $n=1$, $a_n = a_1 = -1$ by given and

$$n^2 - 2n = 1^2 - 2(1) = -1.$$

So $a_n = n^2 - 2n$ for $n=1$.

IND. STEP: Assume $a_k = k^2 - 2k$ for some $k \in \mathbb{N}$.

By the sequence defn,

$$\begin{aligned}
a_{k+1} &= a_k + 2k - 1 \\
&= (k^2 - 2k) + 2k - 1 \quad \text{(by the ind hyp)} \\
&= k^2 + 2k \quad -2k - 1 \quad \text{(reorganizing)} \\
&= k^2 + 2k + 1 - 1 - 2k - 1 \\
&= (k+1)^2 - 2k - 2 \\
&= (k+1)^2 - 2(k+1).
\end{aligned}$$

Thus, $a_{k+1} = (k+1)^2 - 2(k+1)$.

Hence, $a_n = n^2 - 2n$, $\forall n \in \mathbb{N}$ //

6. (16 pts) Let a and b be integers.
Clearly giving a well structure proof and using the precise definitions of even and odd:
Prove a is even and b is even **if and only if** $a + ab + b$ is even.
(Hint: Try indirect methods and cases for one of the directions.)

pf| We must show ① a even AND b even $\Rightarrow$ $a+ab+b$ is even.
② $a+ab+b$ even $\Rightarrow$ a even AND b even.

① Let a and b be even.
By def'n of even, $a=2k$ and $b=2\ell$ for some $k,\ell \in \mathbb{Z}$.
By substitution, $a+ab+b = 2k+2k(2\ell)+2\ell = 2(k+2k\ell+\ell)$.
Since $k+2k\ell+\ell \in \mathbb{Z}$, $a+ab+b$ is even.

② We prove the contrapositive.
Let a be odd <u>OR</u> b be odd.
We consider the possible cases:
ⓘ If a is odd (no assumption on b), then
$a = 2k+1$ for some $k \in \mathbb{Z}$.
By substitution, $a+ab+b = (2k+1)+(2k+1)b+b$
$= 2k+1+2kb+2b$
$= 2(k+kb+b)+1$.
Since $k+kb+b \in \mathbb{Z}$, $a+ab+b$ is odd.
ⓘⓘ Similarly, if b is odd (no assumption on a), then
$b=2\ell+1$ for some $\ell \in \mathbb{Z}$
By substitution, $a+ab+b = a+a(2\ell+1)+(2\ell+1)$
$= 2a\ell+2a+2\ell+1 = 2(a\ell+a+\ell)+1$
Since $a\ell+a+\ell \in \mathbb{Z}$, $a+ab+b$ is odd.
In all cases, $a+ab+b$ is odd. //