1. Find a specific counterexample to each of the following statements:

(a) (5 pts) Let $f(x) = 2x$. If A and B are subsets of $\mathbb{R}$ such that $f : A \to B$, then f is a bijection. (Hint: You'll give sets A and B).

$$\boxed{\begin{array}{l} A = \{1,2\} \\ B = \{1,2,3,4\} \end{array}}$$

$f : A \to B$ because $f(1) = 2$

$f(2) = 4$

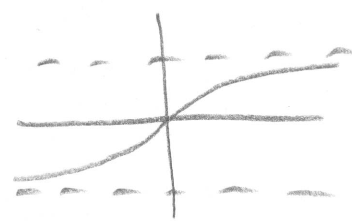
f is one-to-one, but

f is <u>NOT ONTO</u>.

(b) (5 pts) Every strictly monotone function from $\mathbb{R}$ to $\mathbb{R}$ is unbounded.
(If you can't come up with a specific function, draw the graph of a function for partial credit)

NEED A FUNCTION THAT IS STRICTLY MONOTONE AND BOUNDED.

$$\boxed{f(x) = \arctan(x)}$$



(c) (5 pts) $\forall a, b, c \in \mathbb{N}$, if there exist $x, y \in \mathbb{Z}$ such that $ax + by = c$, then $\gcd(a, b) = c$.

WE KNOW THAT $\gcd(a,b)$ <u>divides</u> c, BUT THAT DOESN'T MEAN IT EQUALS c.

$a = 2, b = 3, c = 7$ $\quad ax + by = c$ is possible with $\underline{x = 2, y = 1}$

$2(2) + 3(1) = 7$ ✓

BUT $\gcd(a,b) = \gcd(2,3) = 1 \neq 7$.

2. (6 pts) Determine the coefficient of x^9 in the expansion of $(x - 1)^{12}$.

$$(x-1)^{12} = \binom{12}{0} x^{12} (-1)^0 + \binom{12}{1} x^{11} (-1)^1 + \cdots + \underbrace{\binom{12}{3} x^9 (-1)^3}_{} + \cdots + \binom{12}{12} x^0 (-1)^{12}$$

$$\binom{12}{3} = \frac{12!}{3!\,9!} = \frac{12 \cdot 11 \cdot 10 \cdot \cancel{9} \cdot \cancel{8} \cdot \cancel{7} \cdots \cancel{1}}{3 \cdot 2 \cdot 1 \quad \cancel{9} \cdot \cancel{8} \cdot \cancel{7} \cdots \cancel{1}} = 2 \cdot 11 \cdot 10 = 220$$

$(-1)^3 = -1$ $\quad$ So $\quad \binom{12}{3} x^9 (-1)^3 = \underbrace{\boxed{-220}}_{\text{coefficient}} x^9$

3. (12 pts) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions. Suppose there exists a constant, a, such that the following relationship holds:

$$|f(x) - f(y)| \geq |g(x+a) - g(y+a)| \text{ for all } x, y \in \mathbb{R}.$$

Prove that if g is injective, then f is injective.

pf) Assume g is injective.

Let $x_1, x_2 \in \mathbb{R}$ such that $\boxed{f(x_1) = f(x_2).}$

By the given relationship (with $x = x_1$ and $y = x_2$),

$$|f(x_1) - f(x_2)| \geq |g(x_1+a) - g(x_2+a)|.$$

Since $f(x_1) = f(x_2)$, $|f(x_1) - f(x_2)| = 0$. Thus,

$0 \geq |g(x_1+a) - g(x_2+a)| \geq 0$, so $g(x_1+a) - g(x_2+a) = 0$.

So $g(x_1+a) = g(x_2+a)$. Since g is injective, $x_1 + a = x_2 + a$.

Subtracting a, $\boxed{x_1 = x_2}$ Hence, f is injective. //

4. (10 pts) Let a, b, and c be nonzero integers. Prove that if $a \mid b$ and $d = \gcd(b, c)$, then $ad \mid bc$

pf) Assume $a|b$ and $d = \gcd(b, c)$.

By def'n, $b = ak$ for some $k \in \mathbb{Z}$.

By def'n of gcd, $d = \gcd(b,c)$ implies $d|b$, $d|c$ and d is the largest such common divisor.

Since $d|c$, $c = d\ell$ for some $\ell \in \mathbb{Z}$.

By substitution, $bc = (ak)(d\ell) = ad\underbrace{(k\ell)}_{\text{integer}}$.

Thus, $ad \mid bc$ //

5. Assume $f : (0, \infty) \to \mathbb{R}$ is a function that satisfies the relationship

$$f\left(\frac{a}{b}\right) = f(a) - f(b) \text{ for all } a, b \in (0, \infty).$$

(Hint: For each part below you will be making specific substitutions in for a and b. You are **given** the fact that anything from $(0, \infty)$ can be substituted in for a and b).

(a) (5 pts) Prove that $f(1) = 0$.

Letting $a = 1$ and $b = 1$ gives

$$f\left(\frac{1}{1}\right) = f(1) - f(1) = 0$$

So $\boxed{f(1) = 0}$.

(b) (12 pts) Prove that if $f(t) > 0$ for all real numbers $t > 1$, then f is an increasing function on the set $(0, \infty)$.

pf) Assume $f(t) > 0$ $\forall t \in \mathbb{R}, t > 1$.

Let $x_1, x_2 \in (0, \infty)$ such that $\boxed{x_1 < x_2}$

Using the given fact with $a = x_2$ and $b = x_1$,

$$f\left(\frac{x_2}{x_1}\right) = f(x_2) - f(x_1).$$

Since $0 < x_1 < x_2$, we have $1 < \frac{x_2}{x_1}$, so

By assumption $f\left(\frac{x_2}{x_1}\right) > 0$.

Thus, $f(x_2) - f(x_1) = f\left(\frac{x_2}{x_1}\right) > 0$.

Hence, $\boxed{f(x_1) < f(x_2)}$ //

REMEMBER TO VERY CLEARLY GIVE THE STRUCTURE, ORDER, AND SPECIFIC JUSTIFICATIONS OF STEPS IN YOUR PROOF.

6. (20 pts) For sets A, B, and C, let $f : A \to B$ and $g : B \to C$ be functions. Define $h = g \circ f : A \to C$. That is, $h(a) = g(f(a))$ for all $a \in A$.

   Prove that if f is a bijection and h is a bijection, then g is a bijection.

Assume f and h are bijection.

① First, we prove g is injective.

Assume $g(b_1) = g(b_2)$ for $b_1, b_2 \in B$

Since f is surjective, there exists $a_1, a_2 \in A$

such that $f(a_1) = b_1$ and $f(a_2) = b_2$.

By substitution, $g(f(a_1)) = g(b_1) = g(b_2) = g(f(a_2))$.

Hence, by def'n of h, $h(a_1) = h(a_2)$.

Since h is injective, $a_1 = a_2$.

And since f is a function, it is well-defined, so

$b_1 = f(a_1) = f(a_2) = b_2$. $\Rightarrow b_1 = b_2$.

② Second, we prove g is surjective.

Let $c \in C$.

Since h is surjective, there exists $a \in A$ such that $h(a) = c$. By def'n of h, $g(f(a)) = c$.

Since f is a function from A to B, $f(a) = b$ for some $b \in B$. Hence, $g(b) = c$ for some $b \in B$. //