# Basic First Order Differential Equation Applications

A *differential equation* is an equation involving derivatives. When doing an applied problem, you should start by labeling all quantities and drawing a figure/diagram. Also identify the dependent and independent variables. Basic differential equation mathematical modeling facts:

| 'rate of change of BLAH' | means $\frac{d(BLAH)}{dt}$ |
|---|---|
| '...is proportional to...' | means '...is a constant multiple of...' |
| match units | units of $\frac{dy}{dt}$ should be $y$-units/$t$-units |

Here is a list of very basic examples which only require these translation facts. See if you can translate these into differential equations (answers on the next page).

1. "A city has 160 people. The rate of population growth is a constant 10 people per year."

2. "A city has 3000 people. The rate of population growth is proportional to the population size."

3. "A city has 2100 people. The rate of population growth is the sum of two rates: the birth/death rate and immigration rate. The birth/death rate is proportional to the population size. The rate due to immigration is a constant 100 people/year into the city."

4. "In a 1000 person town, exactly 200 people currently have a certain cold virus. Assume the rate at which the virus is spreading is proportional to the number of people that do NOT have the virus."

5. "A coffee starting at a temperature of 180°F is in a room that is a constant 70°F. The rate of cooling is proportional to the difference between the coffee temperature and the room temperature."

6. "You start with $1000 in your retirement account. You invest an average of 5000 dollars/year. In addition, your money earns interest at a rate of 6% anually, compounded continuously."

   NOTE: "6% annually, compounded continuously" is the bank's way of saying the rate of change due to interest is proportional to the amount of money in the account (with proportionality constant 0.06).

7. "A 100 liter vat full of water contains 75 kilograms of salt . Pure water is pumped in at 5 liters/min and well mixed. Water is also coming out of the bottom of the vat at 5 liters/min."

8. "A 100 liter vat full of water contains 75 kilograms of salt . Salt water with a concentration of 3 kg/L is pumped in at 5 liters/min and well mixed. Water is also coming out of the bottom of the vat at 5 liters/min."

9. "A 100 liter vat currently contains 7 liters of water with 75 kilograms of salt. Salt water with a concentration of 3 kg/L is pumped in at 5 liters/min and well mixed. Water is also coming out of the bottom of the vat at 1 liters/min. (Note: The vat is gaining a constant $5 - 1 = 4$ liters/min of water!)"

10. "A tall circular cylindrical piece of ice (an ice core) with initial radius of 4 inches is accidentally left out and starts to melt. Assume the height remains a constant 200 inches and that the rate of change of volume is proportional to the surface area around the cylinder (i.e. ignore the top/bottom)."

1. *Translation*: If $P(t) =$ the pop. at time $t$, then $\dfrac{dP}{dt} = 10$ with $P(0) = 160$. (Constant growth)

2. *Translation*: If $P(t) =$ the pop. at time $t$, then $\dfrac{dP}{dt} = kP$ with $P(0) = 3000$.
   (This is called natural growth)

3. *Translation*: If $P(t) =$ the pop. at time $t$, then $\dfrac{dP}{dt} = kP + 100$ with $P(0) = 2100$.

4. *Translation*: If $P(t) =$ the number of people that have the virus at time $t$, then
   $1000 - P(t) =$ the number of people that do NOT have the virus at time $t$.
   And we have $\dfrac{dP}{dt} = k(1000 - P)$ with $P(0) = 200$.

5. *Translation*: If $T(t) =$ the temp. of the coffee at time $t$, then $\dfrac{dT}{dt} = k(T - 70)$ with $T(0) = 180$.
   (This is called Newton's Law of Cooling)

6. *Translation*: If $A(t) =$ dollars saved after $t$ years, then $\dfrac{dA}{dt} = 0.06A + 5000$ with $A(0) = 1000$.

7. *Translation*: If $y(t) =$ kilograms of salt after $t$ minutes, then
   Salt IN $= (0 \text{ kg/L})(5 \text{ L/min}) = 0$ kg/min, Salt OUT $= (y/100 \text{ kg/L})(5 \text{ L/min}) = \frac{1}{20} y$ kg/min
   Thus, $\dfrac{dy}{dt} = 0 - \dfrac{1}{20}y$ with $y(0) = 75$.

8. *Translation*: If $y(t) =$ kilograms of salt after $t$ minutes, then
   Salt IN $= (3 \text{ kg/L})(5 \text{ L/min}) = 15$ kg/min, Salt OUT $= (y/100 \text{ kg/L})(5 \text{ L/min}) = \frac{1}{20} y$ kg/min
   Thus, $\dfrac{dy}{dt} = 15 - \dfrac{1}{20}y$ with $y(0) = 75$.

9. *Translation*: First, note that the vat starts with 7 Liters of water and the amount of water in the
   vat is increasing at a constant rate of 4 Liters/min (thus, the equation for the amount of water in
   the vat is a line!). So the amount of water in the vat at time $t$ is given by $7 + 4t$ Liters.
   Now, if $y(t) =$ kilograms of salt after $t$ minutes, then
   Salt IN $= (3 \text{ kg/L})(5 \text{ L/min}) = 15$ kg/min, Salt OUT $= (y/(7+4t) \text{ kg/L})(1 \text{ L/min}) = \frac{y}{7+4t}$ kg/min
   Thus, $\dfrac{dy}{dt} = 15 - \dfrac{y}{7 + 4t}$ with $y(0) = 75$.

10. *Translation*: Let $V(t) =$ the volume of ice. Cylinder surface area (without the top/bottom) is
    given by $SA = 2\pi r h$. And $h$ is a constant 200 inches, so we have $SA = 400\pi r$. The statement of
    the problem translates to $\dfrac{dV}{dt} = k400\pi r$, but it would be better to have everything in terms of $V$
    and $t$ (instead of $r$ which depends on $V$).

    Since the volume of the cylinder is $V = \pi r^2 h = 200\pi r^2$, we have that $r = \sqrt{\dfrac{V}{200\pi}}$. Thus, we get

    $$\dfrac{dV}{dt} = k400\pi \sqrt{\dfrac{V}{200\pi}} = 2k\sqrt{200\pi V} = C\sqrt{V} \text{ with } V(0) = 200\pi(4)^2 = 3200\pi.$$

Various Notes/Comments:

1. **Law of Natural Growth**: "The rate of change of $y$ is proportional to $y$". (The model for unrestricted population growth, radioactive decay, and compound interest)

2. **Newton's Law of Cooling**: "The rate of change of temperature an object is proportional to the difference between the object and the surrounding temperature." Here $k$ is call the cooling constant.

3. **Mixing Problems**: Here we have to find the $\frac{dy}{dt} =$ Rate IN - Rate OUT. The interesting part is the rate OUT, because it depends on the current concentration, *i.e.* always depends on $y$! Here is a fairly standard and full example:
   *Question*: A 100 gallon vat of Brine (salt-water) contains 30 kg of salt diluted in it. Let $y(t)$ be the amount of salt at time $t$. A brine mixture that is 25% salt (units of kg/gallon) is pumped IN at 8 gallons per minute and the mixed vat is emptied OUT at the same rate.
   *Answer*: IN $= 8$ gal/min 0.25 kg/gal $= 2$ kg/min, OUT $= 8$ gal/min $\frac{y(t)}{100}$ kg/gal $= 8y/100$ kg/min
   Thus, $\frac{dy}{dt} = 2 - \frac{8y}{100}$ with $y(0) = 30$.

4. **Force/Motion Problems**: By Newton's second law, $F = ma$, where $F$ is force, $m$ is mass, and $a$ is acceleration. Most applied motion problems proceed as follows:
   Describe all the forces (draw a force diagram) and determine a labeling (what is positive, what is negative). Add up the forces to get the total force, $F$. Then let $v =$ velocity and write the differential equation: $m\dfrac{dv}{dt} = ma = F$. Here are some common scenarios:

   (a) "A falling object near earth's surface ignoring air resistance."
   The only force is due to gravity.
   Thus, $m\dfrac{dv}{dt} = \pm mg$ (use '+' if down is considered increasing distance and '-' if down is considered decreasing distance).

   (b) "A falling object near earth's surface with air resistance."
   i. 'lower velocities': The force due to air resistance is proportional to velocity (in opposite to the direction of the object). If we label down as increasing distances and the object is only going downward (jump from a plane), then we get: $m\dfrac{dv}{dt} = mg - kv$
   ii. 'bigger objects, higher velocities': The force due to air resistance is proportional to the square of velocity (in opposite to the direction of the object). Again, labeling as before (jumping from a plane), then we get $m\dfrac{dv}{dt} = mg - kv^2$.

   (c) "Far away from the Earth's surface": Let $x$ be the distance from the surface of the earth and $R$ be the radius of the Earth. In terms of $x$, the force due to gravity is given by $F = -\frac{mgR^2}{(R+x)^2}$ (law of gravitation). Thus, we get $m\dfrac{dv}{dt} = -\dfrac{mgR^2}{(R+x)^2}$.

   (d) "Object in a liquid": Much is the same in a liquid, except there is a bouyant force as well. You still have resistance which is often modeled as a constant times velocity (same as with air resistance). The bouyant force is the weight of the liquid displaced by the object (let $b$ be the mass of the liquid displaced by the object). In any case, you get a differential equation that looks something like $m\dfrac{dv}{dt} = mg - kv - bg$

**Aside - an extra discussion about 3D motion**

A full discussion of motion requires tools from Math 126 (specifically, vectors, 3D parametric equations, and 3D parametric calculus). We will not discuss 3D motion problems in this course.

However, for your own interest, let me say a few things. In Math 126, you learn to study $x = x(t)$, $y = y(t)$, and $z = z(t)$ separately in terms of time.

In general, we let $v_x(t) = \frac{dx}{dt}$, $v_y(t) = \frac{dy}{dt}$ and $v_z(t) = \frac{dx}{dt}$ be the $x$, $y$, and $z$ velocities.

If $F_x$, $F_y$, and $F_z$ are the $x$, $y$, and $z$ components of the forces acting on an object, then we get the three differential equation

$$m\frac{dv_x}{dt} = F_x \ , \ m\frac{dv_y}{dt} = F_y \ , \ m\frac{dv_z}{dt} = F_z$$

Once we understand this set up, the process is the same as on the previous page.

1. "A falling object near earth's surface ignoring air resistance."
   The only force is due to gravity and it only effects the $z$-component.
   Thus, $m\frac{dv_x}{dt} = 0$ , $m\frac{dv_y}{dt} = 0$ , $m\frac{dv_z}{dt} = -mg$.

2. "A falling object near earth's surface with air resistance." :
   $$m\frac{dv_x}{dt} = -kv_x \ , \ m\frac{dv_y}{dt} = -kv_y \ , \ m\frac{dv_z}{dt} = -mg - kv_z$$

   You would then solve for the functions $v_x(t)$, $v_y(t)$, and $v_z(t)$ separately.