Extra Notes Based on Student Questions

1. Why step functions?
2. How to use them.
3. Laplace Transform
4. Inverse Laplace Transform

+ YOU WILL SEE HW HINTS & ANSWERS FOR HW 9 READ ON...

① WHY STEP FUNCTIONS?

MOTIVATION

Consider $f(t) = \begin{cases} 2, & 0 \le t < 4; \\ 8, & t \ge 4. \end{cases}$

Using the definition,

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^4 e^{-st}\,2\,dt + \int_4^\infty e^{-st}\,8\,dt$$

$$= \left[\frac{2}{-s} e^{-st} \Big|_0^4\right] + \lim_{A\to\infty}\left[-\frac{8}{s} e^{-st}\Big|_4^A\right]$$

$$= -e^{-4s}\frac{2}{s} + \frac{2}{s} + \lim_{A\to\infty}\left[\cancel{-\frac{8}{s}e^{-As}}^{\,0} + e^{-4s}\frac{8}{s}\right]$$

$$= \frac{2}{s} - e^{-4s}\frac{2}{s} + e^{-4s}\frac{8}{s}$$

$$= \boxed{\frac{2}{s} + e^{-4s}\frac{6}{s}}$$

We do NOT want to have to do this every time we have discontinuous forcing. So we standardize with unit step functions.

$u_c(t) = \begin{cases} 0, & t < c; \\ 1, & t \ge c. \end{cases}$

$$f(t) = 2 + (\overset{\text{NEW}}{8} - \overset{\text{OLD}}{2})\,u_4(t) = \underbrace{2 + 6u_4(t)}_{} = \begin{cases} 2, & 0 \le t < 4; \\ 8, & t \ge 4 \end{cases}$$

IF WE CAN WRITE THIS THEN WE CAN USE WHAT WE ALREADY DERIVED!

MUCH FASTER

$$\mathcal{L}\{2 + 6u_4(t)\} = 2\mathcal{L}\{1\} + 6\mathcal{L}\{u_4(t)\}$$

$$= \boxed{\frac{2}{s} + e^{-4s}\frac{6}{s}}$$ ← SAME

② HOW TO USE STEP FUNCTIONS

Ex) $f(t) = \begin{cases} A, & t < c_1; \\ B, & c_1 \le t < c_2; \\ C, & t \ge c_2. \end{cases}$

$$= A + (\overset{\text{NEW}}{B} - \overset{\text{OLD}}{A})\, u_{c_1}(t) + (\overset{\text{NEW}}{C} - \overset{\text{OLD}}{B})\, u_{c_2}(t)$$

NOTE: $\cdot\ A + (B - A) = B$ ← for $c_1 \le t < c_2$;

$\cdot\ A + (B-A) + (C-B) = C$ ← for $c_2 \le t$.

COOL!

EXAMPLES WRITE THESE IN TERMS OF STEP FUNCTIONS.

① $g(t) = \begin{cases} 4, & t < 1; \\ t, & 1 \le t < 10; \\ t + \sin(t-10), & t \ge 10 \end{cases}$

ANSWER

$$g(t) = 4 + (\overset{\text{NEW}}{t} - \overset{\text{OLD}}{4})\, u_1(t) + (\overbrace{t + \sin(t-10)}^{\text{NEW}} - \overset{\text{OLD}}{t})\, u_{10}(t)$$

$$= 4 + (t-4)\, u_1(t) + (\sin(t-10))\, u_{10}(t)$$

LIKE HW 9/9

② $h(t) = \begin{cases} t, & t < 5; \\ 7, & 5 \le t < 9; \\ 0, & 9 \le t. \end{cases}$

ANSWER

$$h(t) = t + (\overset{\text{NEW}}{7} - \overset{\text{OLD}}{t})\, u_5(t) + (\overset{\text{NEW}}{0} - \overset{\text{OLD}}{7})\, u_9(t)$$

$$= t + (7-t)\, u_5(t) - 7u_9(t).$$

HW 9/13

③ $f(t) = \begin{cases} 0, & t < 4; \\ t^2 - 8t + 32, & t \ge 4 \end{cases} = 0 + (t^2 - 8t + 32)\, u_4(t)$ Not $(t-4)^2 + 16$

$\mathcal{L}\{f(t)\} =$

③ LAPLACE TRANSFORM OF STEP FUNCTIONS!

In class, we derived

$$\mathcal{L}\{u_c(t)f(t-c)\} = \int_0^\infty e^{-st} u_c(t) f(t-c)\,dt$$

$$= \int_c^\infty e^{-st} f(t-c)\,dt \qquad u = t-c, \; du = dt$$

$$= \int_0^\infty e^{-s(u+c)} f(u)\,du$$

$$= e^{-cs}\int_0^\infty e^{-su} f(u)\,du$$

$$= e^{-cs}\int_0^\infty e^{-st} f(t)\,dt$$

VARIABLE DOESN'T MATTER

$$= e^{-cs}\mathcal{L}\{f(t)\}$$

REPLACE "t" WITH "t-c"

$$\Rightarrow \mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$$

REPLACE "t" WITH "t+c"

TO USE THIS SEE THE NEXT PAGES FOR EXAMPLES.

HW 9/12 PART OF QUESTION

SHIFT! REPLACE "t" WITH "t+3"

$$\mathcal{L}\{(t-4)u_3(t)\} = e^{-3s}\mathcal{L}\{(t+3)-4\}$$

$$= e^{-3s}\mathcal{L}\{t-1\}$$

$$= \boxed{e^{-3s}\left(\frac{1}{s^2}-\frac{1}{s}\right)}$$

HW 9/13 $f(t) = \begin{cases} 0, & t<4; \\ t^2-8t+32, & t\geq 4. \end{cases}$ NOTE: $t^2-8t+32 = (t-4)^2+16$ COMPLETE SQUARE!

$$= 0 + ((t-4)^2+16)u_4(t)$$

SHIFT! REPLACE "t" BY "t+4"

$$\mathcal{L}\{((t-4)^2+16)u_4(t)\} = e^{-4s}\mathcal{L}\{t^2+16\}$$

$$= \boxed{e^{-4s}\left(\frac{2}{s^3}+\frac{16}{s}\right)}$$

MORE EXAMPLES

WHEN THIS COMES OUT WE MUST SHIFT t!

$$\mathcal{L}\{u_c(t)g(t)\} = e^{-cs}\mathcal{L}\{g(t+c)\}$$

MANY EXAMPLES

SHIFT

① $\mathcal{L}\{u_2(t)\, t\} = e^{-2s}\mathcal{L}\{t+2\}$

$= \boxed{e^{-2s}\left(\frac{1}{s^2}+\frac{2}{s}\right)}$

LIKE HW 9/10(b)

SHIFT

② $\mathcal{L}\{u_7(t)\, e^{3t}\} = e^{-7s}\mathcal{L}\{e^{3(t+7)}\}$

$= e^{-7s}\mathcal{L}\{e^{3t}e^{21}\}$ A NUMBER

$= e^{21}e^{-7s}\mathcal{L}\{e^{3t}\}$

$= \boxed{e^{21}e^{-7s}\frac{1}{s-3}}$

SHIFT

③ $\mathcal{L}\{u_3(t)\sin(6(t-3))\} = e^{-3s}\mathcal{L}\{\sin(6t)\}$

$= \boxed{e^{-3s}\frac{6}{s^2+36}}$

LIKE HW 9/10(c)

SHIFT

④ $\mathcal{L}\{u_4(t)\sin(2t)\} = e^{-4s}\mathcal{L}\{\sin(2(t+4))\}$

$= e^{-4s}\mathcal{L}\{\sin(2t+8)\}$ NUMBER

$= e^{-4s}\{\sin(2t)\cos(8)+\sin(8)\cos(2t)\}$

$= \boxed{e^{-4s}\left(\cos(8)\frac{2}{s^2+4}+\sin(8)\frac{s}{s^2+4}\right)}$

DARN, THIS IS MESSY! NEED AN IDENTITY

$$\sin(A+B) = \sin(A)\cos(B) + \sin(B)\cos(A)$$

(4) INVERSE LAPLACE TRANSFORM OF STEP FUNCTIONS

SINCE $\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} \underbrace{\mathcal{L}\{f(t)\}}_{F(s)}$

NEED TO SHIFT BACK: "t" TO "$t-c$"

$$\mathcal{L}^{-1}\{e^{-cs} F(s)\} = u_c(t)\, \mathcal{L}^{-1}\{F(s)\}\Big|_{t \to t-c}$$

Examples

(1) $\mathcal{L}^{-1}\left\{e^{-4s}\left(\frac{s}{s^2+9} + \frac{1}{s^2}\right)\right\} = u_4(t)\, \mathcal{L}^{-1}\left\{\frac{s}{s^2+9} + \frac{1}{s^2}\right\}\Big|_{t \to t-4}$

$$= u_4(t)\left(\cos(3t) + t\Big|_{t \to t-4}\right)$$

$$= \boxed{u_4(t)\left(\cos(3(t-4)) + t - 4\right)}$$

HW 9/15

(2) $e^{-s} \frac{s-2}{(s^2-4s+3)} \Rightarrow \frac{s-2}{(s-1)(s-3)} = \frac{A}{s-1} + \frac{B}{s-3}$

$A = \frac{1-2}{1-3} = \frac{-1}{-2} = \frac{1}{2}$

$B = \frac{3-2}{3-1} = \frac{1}{2}$

$$\mathcal{L}^{-1}\left\{e^{-s}\left(\frac{1/2}{s-1} + \frac{1/2}{s-3}\right)\right\} = u_1(t)\, \mathcal{L}^{-1}\left\{\frac{1/2}{s-1} + \frac{1/2}{s-3}\right\}_{t \to t-1}$$

$$= u_1(t)\left(\frac{1}{2}e^{t} + \frac{1}{2}e^{3t}\Big|_{t \to t-1}\right)$$

$$\boxed{= u_1(t)\left(\frac{1}{2}e^{t-1} + \frac{1}{2}e^{3(t-1)}\right)}$$

HW 9/17

PUTTING IT ALL TOGETHER

$$y'' + 4y = e^{-3t} u_1(t), \quad y(0) = 0, \; y'(0) = 0$$

$$s^2 \mathcal{L}\{y\} - s y(0) - y'(0) + 4\mathcal{L}\{y\} = \mathcal{L}\{e^{-3t} u_1(t)\}$$

SHIFT!!!

$$(s^2+4)\mathcal{L}\{y\} = e^{-s}\mathcal{L}\{e^{-3(t+1)}\}$$

NUMBER

$$= e^{-s}\mathcal{L}\{e^{-3t}e^{-3}\}$$

$$\Rightarrow (s^2+4)\mathcal{L}\{y\} = e^{-3}e^{-s}\frac{1}{s+3}$$

$$\mathcal{L}\{y\} = e^{-3}e^{-s}\frac{1}{(s+3)(s^2+4)}$$

$$\frac{1}{(s+3)(s^2+4)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+4}$$

$$\Rightarrow 1 = A(s^2+4) + (Bs+C)(s+3) \;\rightarrow\; s=-3 \Rightarrow 1 = 13A$$

$$\boxed{A = 1/13}$$

$$1 = As^2 + 4A + Bs^2 + 3Bs + Cs + 3C$$

$$\Rightarrow 1 = (A+B)s^2 + (3B+C)s + (4A+3C)$$

$\boxed{A = 1/13}$ AND $A + B = 0 \Rightarrow \boxed{B = -1/13}$

AND $3B + C = 0 \Rightarrow \boxed{C = -3B = 3/13}$

$$y = \mathcal{L}^{-1}\left\{e^{-3}e^{-s}\left(\frac{1/3}{s+3} + \frac{-1/13\, s}{s^2+4} + \frac{\frac{3}{13}}{s^2+4}\right)\right\}$$

SHIFT!

$$= e^{-3}u_1(t)\,\mathcal{L}^{-1}\left\{\frac{1/3}{s+3} - \frac{1}{13}\frac{s}{s^2+4} + \frac{3}{13}\frac{1}{2}\frac{2}{s^2+4}\right\}\Bigg|_{t\to t-1}$$

$$= e^{-3}u_1(t)\left(\frac{1}{3}e^{-3t} - \frac{1}{13}\cos(2t) + \frac{3}{26}\sin(2t)\Big|_{t\to t-1}\right)$$

$$= e^{-3}u_1(t)\left(\frac{1}{3}e^{-3(t-1)} - \frac{1}{13}\cos(2(t-1)) + \frac{3}{26}\sin(2(t-1))\right)$$