1. (12 pts)

(a) Find the explicit solution to the initial value problem $\dfrac{dy}{dx} = e^{-3y}\sin(x)$ with $y(0) = 0$.

$$\int e^{3y}\,dy = \int \sin(x)\,dx$$

$$\frac{1}{3}e^{3y} = -\cos(x) + C_1$$

$$C_2 = 3C_1$$

$$e^{3y} = -3\cos(x) + C_2$$

$$3y = \ln(-3\cos(x) + C_2)$$

$$y = \frac{1}{3}\ln(-3\cos(x) + C)$$

$$y(0) = 0 \Rightarrow 0 = \frac{1}{3}\ln(-3\underbrace{\cos(0)}_{1} + C)$$

$$\Rightarrow -3 + C = 1 \Rightarrow C = 4$$

$$\boxed{y = \frac{1}{3}\ln(4 - 3\cos(x))}$$

(b) Find the explicit solution to the initial value problem $t\dfrac{dy}{dt} + \dfrac{1}{2}y = 3t - \sqrt{t}$ with $y(1) = 10$.

$$\frac{dy}{dt} + \frac{1}{2}\frac{1}{t}y = 3 - \frac{\sqrt{t}}{t} = 3 - \frac{1}{\sqrt{t}}$$

choose $C_0 = 0$

$$\int \frac{1}{2}\frac{1}{t}\,dt = \frac{1}{2}\ln(t) + C_0$$

$$\mu(t) = e^{\frac{1}{2}\ln(t)} = e^{\ln(t^{\frac{1}{2}})} = t^{\frac{1}{2}} = \sqrt{t}$$

$$\frac{d}{dt}(\sqrt{t}\,y) = 3\sqrt{t} - 1$$

$$\sqrt{t}\,y = 2t^{3/2} - t + C_1$$

$$y = 2\frac{t^{3/2}}{\sqrt{t}} - \frac{t}{\sqrt{t}} + \frac{C}{\sqrt{t}}$$

$$y = 2t - \sqrt{t} + \frac{C}{\sqrt{t}}$$

$$y(1) = 10$$

$$\Rightarrow 10 = 2 - 1 + C$$

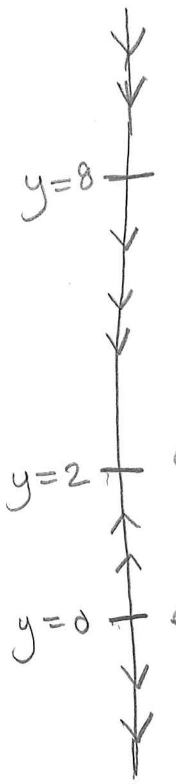
$$C = 9$$

$$\boxed{y = 2t - \sqrt{t} + \frac{9}{\sqrt{t}}}$$

2. (8 pts) For all the parts below, consider $\frac{dy}{dt} = y(8 - y^3)(y - 8)^2$

(a) Find and classify the equilibrium solutions (Show your work and *clearly* label each answer as Stable, Unstable or Semistable).

$$y(8-y^3)(y-8)^2 \stackrel{?}{=} 0 \Rightarrow \begin{array}{l} y = 0 \\ \text{OR} \\ 8 - y^3 = 0 \Rightarrow y = 2 \\ \text{OR} \\ y - 8 = 0 \Rightarrow y = 8 \end{array}$$



$\frac{dy}{dt} = (+)(-)(+) = -$

← SEMISTABLE

$\frac{dy}{dt} = (+)(-)(+) = -$

← STABLE

$\frac{dy}{dt} = (+)(+)(+) = +$

← UNSTABLE

$\frac{dy}{dt} = (-)(+)(+) = -$

$y(t) = 8$	SEMISTABLE
$y(t) = 2$	STABLE
$y(t) = 0$	UNSTABLE

(b) If $y = f(t)$ is a solution to the differential equation above with initial condition $f(10) = 7.5$, then what is $\lim_{t\to\infty} f(t)$? (Explain)

SINCE $y_0 = 7.5$ IS BETWEEN 2 AND 8 AND ALL INITIAL CONDITIONS BETWEEN 2 AND 8 LEAD TO SOL'NS THAT APPROACH 2, WE HAVE $\boxed{\lim_{t\to\infty} f(t) = 2}$

3. (10 pts) Clara has \$500 dollars right now in a savings account that earns about 5% annual interest (compounded continuously). She regularly makes small deposits into the account throughout the year but increases this amount each year (deposits totaling about \$100/yr after the first year, about \$200/yr after the second year the second, \$300 after the third, and so on). That is, assume she deposits money at rate of about $100t$ dollars/year, where t is the number of year. A differential equation modeling this scenario is

$$\frac{dy}{dt} = 0.05y + 100t.$$

Solve this equation and determine how money Clara will have in the account in 30 years. (Round your answer to the nearest dollar).

$\frac{dy}{dt} - 0.05y = 100t$ $\qquad \mu(t) = e^{-0.05t}$

$\frac{d}{dt}\left(e^{-0.05t} y\right) = 100te^{-0.05t}$ $\qquad u = 100t \quad dv = e^{-0.05t}\,dt$

$du = 100\,dt \quad v = \frac{-1}{0.05} e^{-0.05t}$

$\uparrow$

-20

$e^{-0.05t} y = \int 100te^{-0.05t}\,dt$

$= -20 \cdot 100t\,e^{-0.05t} - \int -20e^{-0.05t} \cdot 100\,dt$

$\left(e^{-0.05t} y = -2000te^{-0.05t} + \underbrace{\frac{2000}{-0.05}}_{-40000} e^{-0.05t} + C\right) e^{0.05t}$

$y = -2000t - 40000 + Ce^{0.05t}$

$y(0) = 500 \Rightarrow \quad -40000 + C = 500 \Rightarrow \quad C = 40{,}500$

$y(30) = -2000(30) - 40000 + 40500\,e^{0.05(30)}$

$\boxed{\approx \$81{,}508}$

4. (10 pts)

(a) Consider $\frac{dy}{dt} = 2y^2 - 4t$ and $y(0) = 1$.
Use Euler's method with stepsize $h = \frac{1}{2}$ to approximate the value of $y(1)$.

$t_0 = 0, \quad y_0 = 1$

$y_1 = y_0 + f(t_0, y_0)h = 1 + (2(1)^2 - 4(0))\frac{1}{2} = 1 + 2 \cdot \frac{1}{2} = 2, \quad t_1 = \frac{1}{2}$

$y_2 = y_1 + f(t_1, y_1)h = 2 + (2(2)^2 - 4(\frac{1}{2}))\frac{1}{2} = 2 + (8-2)\frac{1}{2} = 5, \quad t_2 = 1$

So $\boxed{y(1) \approx 5}$

(b) You are asked to study~~ing~~ the population of a small town that is having a problem with people moving away. Currently there are 10,000 people in the town. About 2% of the population gives birth to a new baby each year (so $0.02P$ is the rate at which people are added each year). And people are leaving the town at a rate of K people/year (where K is a positive constant). Assume these are the only two ways the population is changing. Let $P(t)$ be the population in t years.

i. Write down the differential equation and initial condition for this description

$$\frac{dP}{dt} = 0.02P - K$$

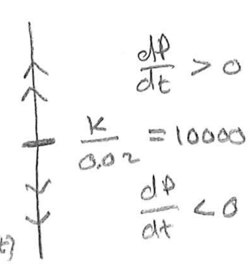
$$P(0) = 10,000$$

ii. For some values of K the population will decrease (and eventually the town will be empty), but for other values of K the population will not decrease. Without solving, find the biggest K can be without causing the population to decrease. Explain your answer.

EQUILIBRIUM $\Rightarrow$ $0.02P - K = 0 \Rightarrow P = \frac{K}{0.02}$

IF EQUILIBRIUM WAS 10,000, THEN POPULATION WILL STAY THE SAME.

WANT $\frac{K}{0.02} \leq 10,000 \Rightarrow \boxed{K \leq 200 \ \frac{\text{people}}{\text{year}}}$

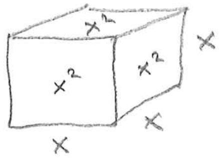


NEED 200 or FEWER PEOPLE TO LEAVE EACH YEAR (or ELSE POP. WILL DECREASE)

5. (10 pts) These are short answer questions. Set-up the differential equation and initial conditions and put a box around your answer. DO NOT SOLVE!

(a) An ice cube is in the shape of a perfect cube with volume 1000 in^3. Label the length, height and width of each side x inches. Assume the ice melts (loses volume) at a rate proportional to the surface area of the entire cube with proportionality constant $k = 0.1$. Write down the differential equation and initial condition for the volume $V = V(t)$ of the ice cube.

(Write it so that the only variables are $\frac{dV}{dt}$ and V).



SURFACE AREA $= S = 6x^2$

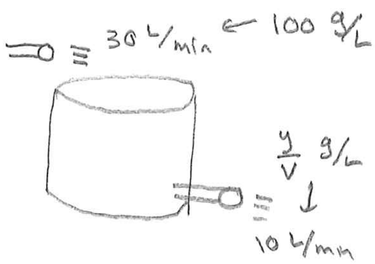
VOLUME $= V = x^3 \Rightarrow x = V^{1/3}$

BY ASSUMPTION, $\frac{dV}{dt} = -0.1 \cdot S = -0.1 \cdot 6x^2$

$$\Rightarrow \boxed{\frac{dV}{dt} = -0.6\, V^{2/3} \qquad V(0) = 1000}$$

(b) A 500-liter tank currently contains 100-liters of water in which 30 grams of salt have been dissolved. Saltwater with a concentration of 100 grams/liter is pumped in at 30 liters/minute and the well mixed saltwater solution is pumped out at 10 liters/minute (thus the volume of the tank is increasing by 30-10 = 20 L/min). Write down the differential equation and initial condition for the amount of salt $y = y(t)$ in grams in the vat after t minutes.

(The only variables in the problem should be $\frac{dy}{dt}$, y, and t).



30 L/min − 10 L/min

VOLUME OF WATER IN TANK $= V = 100 + 20t$

RATE IN $= \left(\text{concentration in}\right)\left(\text{flow in}\right)$

$= 100 \frac{g}{L} \cdot 30 \frac{L}{min} = 3000\ g/min$

RATE OUT $=$ (Conc. out) (flow out)

$= \frac{y}{V} \frac{g}{L} \cdot 10 \frac{L}{min} = \frac{y}{100+20t}\ 10$

$= \frac{y}{10+2t}\ g/min$

$$\boxed{\frac{dy}{dt} = 3000 - \frac{y}{10+2t} \qquad y(0) = 30}$$