Skills Review: Integration By Parts

If you integrate both sides of the product rule and rearrange, then you get the integration by parts formula:

$$\int u dv = uv - \int v du.$$

The method involves choosing u and dv, computing du and v, and using the formula. See the last page of this review for some fast ways to use this formula.

Before we get started, let me remind you of the following integrals (these only require substitution):

Rule	Example
$\int e^{\alpha t}\, dt = \frac{1}{\alpha} e^{\alpha t} + C$	$\int e^{5t}\, dt = \frac{1}{5} e^{5t} + C$
$\int \cos(\alpha t)\, dt = \frac{1}{\alpha} \sin(\alpha t) + C$	$\int \cos(3t)\, dt = \frac{1}{3} \sin(3t) + C$
$\int \sin(\alpha t)\, dt = -\frac{1}{\alpha} \cos(\alpha t) + C$	$\int \sin\left(\frac{1}{4}t\right)\, dt = -4\cos\left(\frac{1}{4}t\right) + C$

Now we can begin. Here are a few classic examples of integration by parts, try them out and see if you can get the given answer (answers are on the right). These should be easy exercises for you, come ask in office hours if they are not and I can give you a refresher:

1. $\int te^{2t}\, dt.$ $\qquad = \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + C.$
2. $\int t\sin(3t)\, dt.$ $\qquad = -\frac{1}{3}t\cos(3t) + \frac{1}{9}\sin(3t) + C.$
3. $\int \ln(t)\, dt.$ $\qquad = t\ln(t) - t + C.$
4. $\int t^2 e^{5t}\, dt.$ $\qquad = \frac{1}{5}t^2e^{5t} - \frac{2}{25}te^{5t} + \frac{2}{125}e^{5t} + C.$

For what we are doing in this class, we need to be able to do the following types of problems:

- $\int te^{\alpha t}\, dt = \frac{1}{\alpha}te^{\alpha t} - \frac{1}{\alpha^2}e^{\alpha t} + C.$
- $\int t^2e^{\alpha t}\, dt = \frac{1}{\alpha}t^2e^{\alpha t} - \frac{2}{\alpha^2}te^{\alpha t} + \frac{2}{\alpha^3}e^{\alpha t} + C.$
- $\int t^3e^{\alpha t}\, dt = \frac{1}{\alpha}t^3e^{\alpha t} - \frac{3}{\alpha^2}t^2e^{\alpha t} + \frac{6}{\alpha^3}te^{\alpha t} - \frac{6}{\alpha^4}e^{\alpha t} + C.$

Never Ending Integration by Parts (and how to end it)
It also is important in this class that we can integrate the following examples (These require integration by parts twice and a 'trick'). Try these:

1. $\int e^t \cos(3t)\, dt$. Answer: $\frac{1}{10}e^t\cos(3t) + \frac{3}{10}e^t\sin(3t) + C$.

2. $\int e^{3t}\sin(4t)\, dt$. Answer: $-\frac{4}{25}e^{3t}\cos(4t) + \frac{3}{25}e^{3t}\sin(4t) + C$.

The general result is (you can quote these):

- $\int e^{\alpha t}\sin(\beta t)\, dt = -\frac{\beta}{\alpha^2+\beta^2}e^{\alpha t}\cos(\beta t) + \frac{\alpha}{\alpha^2+\beta^2}e^{\alpha t}\sin(\beta t) + C$
- $\int e^{\alpha t}\cos(\beta t)\, dt = \frac{\alpha}{\alpha^2+\beta^2}e^{\alpha t}\cos(\beta t) + \frac{\beta}{\alpha^2+\beta^2}e^{\alpha t}\sin(\beta t) + C$

Here is the general proof of one of these formula. Note that we use integration by parts twice, then get all the integrals on one side by adding (that is the key to ending this seemingly never ending integration by parts):

$$\begin{aligned}\int e^{\alpha t}\sin(\beta t)\, dt &= -\frac{1}{\beta}e^{\alpha t}\cos(\beta t) + \frac{\alpha}{\beta}\int e^{\alpha t}\cos(\beta t)\, dt && \text{by parts: } u = e^{\alpha t}, dv = \sin(\beta t)\\ &= -\frac{1}{\beta}e^{\alpha t}\cos(\beta t) + \frac{\alpha}{\beta^2}e^{\alpha t}\sin(\beta t) - \frac{\alpha^2}{\beta^2}\int e^{\alpha t}\sin(\beta t)\, dt && \text{by parts: } u = e^{\alpha t}, dv = \cos(\beta t)\end{aligned}$$

Adding $\frac{\alpha^2}{\beta^2}\int e^{\alpha t}\sin(\beta t)\, dt$ to both sides yields

$$\left(1+\frac{\alpha^2}{\beta^2}\right)\int e^{\alpha t}\sin(\beta t)\, dt = -\frac{1}{\beta}e^{\alpha t}\cos(\beta t) + \frac{\alpha}{\beta^2}e^{\alpha t}\sin(\beta t)$$

Multiplying both sides by β^2 gives

$$\left(\beta^2+\alpha^2\right)\int e^{\alpha t}\sin(\beta t)\, dt = -\beta e^{\alpha t}\cos(\beta t) + \alpha e^{\alpha t}\sin(\beta t)$$

And dividing by $\alpha^2+\beta^2$ gives

$$\int e^{\alpha t}\sin(\beta t)\, dt = -\frac{\beta}{\alpha^2+\beta^2}e^{\alpha t}\cos(\beta t) + \frac{\alpha}{\alpha^2+\beta^2}e^{\alpha t}\sin(\beta t)$$

The other formula can be done in a nearly identical way.

Different Ways To Use Integration by Parts

Most students write u and dv in the margins or in a table, then compute du and v. That works well and is all you need for this course. However, as you go into later courses (like Math 309), you might want faster and more compact ways to do integration by parts. Here are a couple of methods to try. This is just for your own interest.

1. **Use the formula directly**:
   Instead of making a table in the margins you can do the integration and differentiation "in-line" by directly using the formula. For example:

$$\int 4xe^{3x}\,dx = \int 4x\,d\left(\frac{1}{3}e^{3x}\right) = \frac{4}{3}xe^{3x} - \int \frac{1}{3}e^{3x}d(4x) = \frac{4}{3}xe^{3x} - \frac{4}{9}e^{3x} + C$$

   In the evaluation above, we just replace $u = 4x$, $dv = e^{3x}dx$ in $\int u\,dv = uv - \int v\,du$. Notice in the first step, we integrated to find v and replaced it in the formula to write $dv = d(1/3e^{3x})$. Then in the second step, we used the formula replacing u and v. This is not necessarily better than the table method, but it is something you can do in one line. Try it out and see if you like it.

2. **Tabular Integration by Parts**:
   If you need to do multiple steps of integration, then it might be helpful to make a big table of derivatives and integrals. For example: $\int x^4 \sin(x)\,dx$ will require many steps of integration by parts. The sign of the terms will alternate between positive and negative (because the integration by parts formula starts with a '$+uv$' term, then a '$-\int vdu$' term and it will alternate if you keep going). Rather than do each step separately, we can organize our work as follows:

Sign	u	dv
$+$	x^4	$\sin(x)$
$-$	$4x^3$	$-\cos(x)$
$+$	$12x^2$	$-\sin(x)$
$-$	$24x$	$\cos(x)$
$+$	24	$\sin(x)$
	0	$-\cos(x)$

   Then we multiply u and v at each step with alternating signs (just like we would do if we actually did each step with the formula). The final answer is:

$$\int x^4 \sin(x)\,dx = x^4(-\cos(x)) - 4x^3(-\sin(x)) + 12x^2\cos(x) - 24x\sin(x) + 24(-\cos(x)) + C$$

   Thus,

$$\int x^4 \sin(x)\,dx = -x^4\cos(x) + 4x^3\sin(x) + 12x^2\cos(x) - 24x\sin(x) - 24\cos(x) + C$$