2.2: Separation of Variables

In this section, we consider differential equations of the form

$$\frac{dy}{dx} = f(x)g(y).$$

These equations are called **separable** differential equation because the variables t and y can be factored into a product of separate functions $f(t)$ and $g(y)$ that only involve t and y, respectively.

1. How to solve separable differential equations:

   (a) SET UP: Get $\frac{dy}{dx}$ by itself. Factor/separate the other side into the form $f(x)g(y)$.

   (b) SEPARATE: Divide by $g(y)$ on both sides (and multiply by dx).

   (c) INTEGRATE: Evaluate the two integrals:

   $$\int \frac{1}{g(y)}\, dy = \int f(x)\, dx.$$

   (d) CLEAN UP: Remember to put in a constant of integration on one side. Then solve for y if you can and write your answer in a clean way. (Perhaps re-defining your constant to make the solution look nicer)

   (e) INITIAL CONDITION: Put in your initial conditions to solve for any unknown constants (note that you get one constant from integration).

   (f) CHECK YOUR ANSWER: As always, differentiate your answer and check that it satisfies the differential equation (also double check initial conditions).

2. Important Notes

   - This is a method for solving **first order** differential equations. It works for **linear** and **nonlinear** differential equations. However, it is NOT always possible to do step 1 of the process above (you often can't separate variables).
   - If you can solve for y, then we call the final answer an **explicit** solution. So if you are asked to find the explicit solution, then your final answer will look like $y = y(x) =$ 'an expression only involving x.'
   - If you integrate and get an equation involving x and y, but you cannot solve for y, then we call that equation an **implicit** solution as it defines the relationship between x and y implicitly be an equation (which is a perfectly reasonable final answer, but not as easy to use).
   - To determine the **interval over which an explicit solution is defined** you should

     (a) Look at the final answer: Are there are domain restrictions on the function in your final answer?
     (You can sometimes see these restriction directly from the differential equation; for example you can find when the differential equation will give vertical tangents)

     (b) The interval will contain the x value of the initial condition!

     (c) The interval over which the explicit solution is defined will be the largest interval around the initial condition that satisfies the restrictions from the final answer.

Separable Examples:

1. Find the explicit solution to $\dfrac{dy}{dx} = \dfrac{x}{y^2}$ with $y(0) = 2$.

   Abbreviated Solutions:
   $\displaystyle\int y^2 dy = \int x dx$ becomes $\frac{1}{3}y^3 = \frac{1}{2}x^2 + C$. Thus, $y = \left(\frac{3}{2}x^2 + D\right)^{1/3}$ where $D = 3C$.

   Using the initial condition gives $D = 8$, for a final explicit solution of $y = \left(\dfrac{3}{2}x^2 + 8\right)^{1/3}$.

   Note: This function is defined for all values of x.

2. Find the explicit solution to $\dfrac{dy}{dx} = 6xy^2$ with $y(0) = \frac{1}{12}$.

   Abbreviated Solutions:
   $\displaystyle\int \frac{1}{y^2} dy = \int 6x dx$ becomes $-\frac{1}{y} = 3x^2 + C$. Thus, $y = -\frac{1}{3x^2+C}$.

   Using the initial condition gives $C = -12$, for a final explicit solution of $y = -\dfrac{1}{3x^2 - 12}$.

   Note: This function is undefined at $x = \pm 2$. Since the initial condition was between these values, the interval over which the explicit solution is defined is $-2 < x < 2$.

3. Find the explicit solution to $\dfrac{dy}{dx} = x\cos(x)\cos^2(y)$ with $y(0) = 0$.

   Abbreviated Solutions:
   $\displaystyle\int \frac{1}{\cos^2(y)} dy = \int x\cos(x) dx$ becomes $\displaystyle\int \sec^2(y) dy = x\sin(x) - \int \sin(x) dx$ (identity and by parts).
   Integrating gives $\tan(y) = x\sin(x) + \cos(x) + C$.
   Solving for y gives, $y = \tan^{-1}(x\sin(x) + \cos(x) + C)$.
   Using the initial condition gives $0 = \tan^{-1}(0 + 1 + C)$, so $C = -1$, for a final explicit solution of $y = \tan^{-1}(x\sin(x) + \cos(x) - 1)$. *Note*: This function is defined for all values of x.

4. Find an explicit solution to $\dfrac{dy}{dx} = \dfrac{y}{x^2 - 2x}$ with $y(8) = -\sqrt{3}$.

   Abbreviated Solutions:
   $\displaystyle\int \frac{1}{y} dy = \int \frac{1}{x(x-2)} dx$ becomes $\displaystyle\ln|y| = \int \frac{-1/2}{x} + \frac{1/2}{x-2} dx$ (partial fractions)
   Integrating gives $\ln|y| = -\frac{1}{2}\ln|x| + \frac{1}{2}\ln|x-2| + C$.
   Logarithm rules: $\ln|y| = \ln\left(\frac{1}{\sqrt{x}}\right) + \ln\left(\sqrt{x-2}\right) + C$ becomes $\ln|y| = \ln\left(\sqrt{\frac{x-2}{x}}\right) + C$
   Exponentiating both sides gives $y = \pm e^C\sqrt{\frac{x-2}{x}}$. Letting $D = \pm e^C$, we get a cleaner looking answer of $y = D\sqrt{\frac{x-2}{x}} = D\sqrt{1 - \frac{2}{x}}$.
   Using the initial condition gives $-\sqrt{3} = D\sqrt{1 - \frac{2}{8}} = D\sqrt{3/4} = D\sqrt{3}/2$, so $D = -2$, for a final explicit solution of $y = -2\sqrt{1 - \dfrac{2}{x}}$.
   Note: This function is defined if $1 - \frac{2}{x} \geq 0$, which gives $x \geq 2$. (This function is also defined when $x < 0$, but the given initial condition has a value of x bigger than 2). The interval over which this explicit solution is defined is $x \geq 2$.

Substitution
Sometimes when we encounter a differential equation that is not separable, we can perform a change of variable that makes it separable. The idea is to replace an expression involving y by some other variable v in order to get a differential equation involving $\frac{dv}{dx}$ that is separable. Here is a rough outline (there are more sophisticated substitution methods, but we'll save that for another class):

How to solve differential equations of the form $\dfrac{dy}{dx} = f(x, y)$ using substitution:

1. CHOOSE YOUR SUBSTITUTION: Let $v = g(x, y) =$ 'some expression from $f(x, y)$' that you wish to substitute for (see below for advise on substitution).

2. SET UP: Compute $\frac{dv}{dx} = ???$ (this will require implicit differentiation). Somewhere in this derivative you will see $\frac{dy}{dx}$, replace it with $f(x, y)$ and use your definition of v to get rid of y.

3. HOPE: Hope this set up ended with a new differential equation that only involves v and x and is separable.

4. SOLVE: Solve the separable equation.

5. REPLACE: Replace v in your final answer to get a solution involving x and y.

Notes and Examples: Here are some common choices for v with examples.

- Let $v = ax + by + c$.
  For example: $\dfrac{dy}{dx} = 2x - 5y$ is not separable.
  Let $v = 2x - 5y$.
  Differentiate with respect to x to get $\frac{dv}{dx} = 2 - 5\frac{dy}{dx}$.
  Replacing $\frac{dy}{dx}$ gives $\frac{dv}{dx} = 2 - 5(2x - 5y) = 2 - 5v$. This is separable!
  BUT, this example can also be fairly easily done using integrating factors which we will learn in section 2.1.

- Let $v = \frac{y}{x}$.
  For example: $\dfrac{dy}{dx} = \dfrac{y}{x} - e^{y/x}$ is not separable.
  Let $v = \frac{y}{x}$ and rewrite it as $y = xv$.
  Differentiate with respect to x to get $\frac{dy}{dx} = v + x\frac{dv}{dx}$.
  Replacing $\frac{dy}{dx}$ gives $\frac{y}{x} - e^{y/x} = v + x\frac{dv}{dx}$ which can be written as $v - e^v = v + x\frac{dv}{dx}$.
  This can be simplified to the form $\frac{dv}{dx} = -\frac{e^v}{x}$, which is separable!
  This particular substitution is something you might see in a later course.

- Often trying $v =$ INSIDE function is something to try. For example, if I saw $(x + y)^2$, I might try $v = x + y$.

General Comment: One could investigate this method more and build rules for good choices of v, but we will not focus on this method in this class (you will use it more in Math 309). I simply wanted to show you how to change variables and give you another option to try to make an equation separable. If you are stumped on a problem, this is something to try!