2.3: Modeling with Differential Equations

Some General Comments:
A **mathematical model** is an equation or set of equations that mimic the behavior of some phenomenon under certain assumptions/approximations. Phenomena that contain rates/change can often be modeled with differential equations.
Disclaimer: In forming a mathematical model, we make various assumptions and simplifications. I am never going to claim that these models perfectly fit physical reality. But mathematical modeling is a key component of the following scientific method:

1. We make assumptions (a hypothesis) and form a model.
2. We mathematically analyze the model. (For differential equations, these are the techniques we are learning this quarter).
3. If the model fits the phenomena well, then we have evidence that the assumptions of the model might be valid.
   If the model fits the phenomena poorly, then we learn that some of our assumptions are invalid (so we still learn something).

Concerning this course:
I am not expecting you to be an expert in forming mathematical models. For this course and on exams, I expect that:

1. You understand how to do all the homework! If I give you a problem on the exam that is very similar to a homework problem, you must be prepared to show your understanding.
2. Be able to translate a basic statement involving rates and language like in the homework. See my previous posting on applications for practice (and see homework from 1.1, 2.3, 2.5 and throughout the other assignments).
3. If I give an application on an exam that is completely new and involves language unlike homework, then I will give you the differential equation (you won't have to make up a new model). In these cases, you are expected to be able to read carefully, solve/analyze the equation, and use any given initial conditions.

Case Studies:
On the following pages, I discuss a few basic applications in more detail. Some of this information is for your own interest, but it should help you gain a deeper understanding of these models.
Contents:

Mixing Problem Models

"The rate of change of a substance in a system equals the rate at which it enters the systems minus the rate at which it exits the system".

1. *The model*: $\dfrac{dy}{dt}$ = Rate IN - Rate OUT.

2. *Comment*: There is no physical assumption being made (yet).
   If a substance is entering a system at 5 g/min and leaving at 3 g/min, then the overall rate of change within the system is $5 - 3 = 2$ g/min. That isn't an assumption, it is a fact!

3. *Set Up:*

   (a) Label/Identify: $y = y(t) =$ 'amount of substance at time t'. Identify concentrations and rates INTO the system. Identify concentrations and rates OUT of the system. Identify the initial amount of the substance (initial condition).

   (b) Volume: If the overall volume of the system is changing, find a formula for the volume at time t.

   (c) Rates: Units of dy/dt should be y-units/t-units. Please check your units! Often you are given concentrations and rate information in which case:

   - Rate IN = (concentration coming in)(fluid-in rate). In Math 125, these were constants. In this course, we may give a function $f(t)$ for a changing concentration.
   - Rate OUT = (concentration of system)(fluid-out rate) = $\left(\dfrac{y}{V(t)}\right)$(rate out),
     where $V(t)$ = volume of fluid in the system. Since $y = y(t)$ is the current amount of substance, the fraction $\frac{y}{V(t)}$ gives the current concentration in the system at time t.
     (This does assume the system is well mixed, we always make this assumption in these problems)

4. *Behavior/Solutions*: In Math 125, we only did problems where rate in was constant and volume was constant (in which case we got a separable equation). We can now do problems where rate in and/or volume is not constant. These models often give interesting asymptotic behavior. We often ask about the concentration as $t \to \infty$.

Temperature Models

Newton's Law of Cooling: "The rate of change of temperature an object is proportional to the difference between the object and the surrounding temperature."
I have been told that this model is good for heat transfer through *convection* and it models well the rate of change in temperature between a solid and a fluid/gas. I have also been told it is more accurate when the temperature differences are small.

1. *The model*: $\dfrac{dT}{dt} = k(T - T_s)$, where T_s is temperature of the surroundings and $T = T(t)$ is the temperature of the object.

2. *Comments*: This model is sometimes used when the temperature of the surroundings is variable, in which case $T_s = T_s(t)$ would need to be given and the equation will not be separable. In Math 125, we only discussed cases in which T_s was constant, in which case you get a separable equation.

3. *Constants*: The constant k is called the cooling constant and is dependent on the object, surroundings, and container. For cooling coffee, a k closer to zero would mean you have a mug with better insulation.

4. *General Behavior*: If T_s is constant, there is an equilibrium solution at $T(t) = T_s$ and all starting temperatures (above or below this temperature) will give functions that tend toward T_s as $t \to \infty$.

5. *Solutions*: If T_s is constant, we can separate and solve to get $T(t) = T_s + (T_0 - T_s)e^{kt}$. You should know how I got this!. So we see that this assumption leads to an exponential function for temperature.

Stefan-Boltzmann law: "The rate of change of temperature of an object due to radiation is proportional to the difference between the fourth power of the object's temperature and the fourth power of the surrounding temperature."
I have been told that this model is good for heat transfer through *radiation*. An example is a red hot piece of iron radiating heat. In these scenarios typically the difference in temperature is very large.

1. *The model*: $\dfrac{dT}{dt} = k(T^4 - T_s^4)$, where T_s is temperature of the surroundings and $T = T(t)$ is the temperature of the object. To simplify the problem, in the homework you will assume that T_s is small relative to T and the equation will become $\dfrac{dT}{dt} = kT^4$.

2. *Constants*: The constant k is the cooling constant and it depends on the material of the object and surroundings.

3. *General Behavior*: Similar to Newton's law of cooling in terms of equilibrium and long term behavior.

4. *Solutions*: You'll do this in homework for the simplified version. It is NOT exponential, it is a rational function.

NOTE: To truly study heat transfer, you need the heat equation which you will study in Math 309.

Financial Models

Interest bearing accounts where "interest is compounded continuously"

1. *The No Interest Model*: If you put money into a jar (or under your mattress), then you don't earn any interest. If you approximate that you put in \$500 each year, then the model for the amount of money saved will be 'rate of change due to deposits' = 500. In general, 'the rate due to deposits/payments' = P = the annual amount we pay/deposit.

2. *The Amount with Interest Model*: When a banker says "interest has an annual decimal rate of r, compounded continuously", then they are saying that interest is proportional to the amount in the account. Meaning 'the rate due to interest' = ry.
If we invest a lump sum of money $y(0) = y_0$ into an account paying an annual decimal rate of r, compounded continuously and we NEVER deposit or pay any money, then we get the model $\frac{dy}{dt} = ry$ with a solution of $y(t) = y_0 e^{rt}$ for the amount of money in the account after t years.

3. *Compound Interest AND regular deposits/payment*: Typical in saving money or paying off a debt, there is interest AND regular deposits/payments.
Putting this all together: $\frac{dy}{dt} = ry \pm P$ dollars/year
(Note: The $\pm$ would be positive if the deposits are adding to the account balance and negative if you are making payments that decrease the balance of the account)

4. *Comments*: If you take a finance/accounting class in investments or mortgage, you will find that there are precise formulas that can generate payments schedules to the nearest penny. But these formulas are messy and a bit tedious to use. In comparison, the model above is easy to set up and gives a formula that's easy to use (and it gives a very good approximation). Since personal depositing and payments aren't typically precise, the models above can be used to get very good estimates for interest bearing account balances.

5. *Behavior/Solutions*: Solutions are exponential. So compound interest leads to exponential growth. Put away a bit of money each month with a reasonable interest rate and it can grow to be a very large sum over 20-30 years, that's the power of compound interest! (It's exponential!)

Force/Motion Models

By Newton's second law, $F = ma$, where F is force, m is mass, and a is acceleration.
Since $a = \frac{dv}{dt}$, we get $m\frac{dv}{dt} = F =$ 'the sum of all the forces on the object'.

1. *Labeling*: Always draw a force diagram. Decide if you want upward velocities to be positive or negative (this is your choice!), but then be consistent throughout the rest of the problem.

2. *No air resistance*: "A falling object near earth's surface ignoring air resistance."
   The only force is due to gravity. Thus, $m\dfrac{dv}{dt} = \pm mg$, where $g = 9.8$ m/s$^2 = 32$ ft/s^2. (It is positive or negative depending on our labeling).

3. *Air resistance models*: "A falling object near earth's surface with air resistance."
   (a) The force due to air resistance is called the **drag** force. It is always in the direction *opposite* of velocity.
   (b) One model for air resistance (good for smaller objects) is: "The force due to air resistance is proportional to velocity and in the opposite direction."
   Thus, 'drag force' $= -kv$, where k is positive and is called the drag coefficient.
   If we label so that up is positive, then we get the model $m\dfrac{dv}{dt} = -mg - kv$.
   If we label so that down is positive, then we get the model $m\dfrac{dv}{dt} = mg - kv$. (books often use this labeling when studying terminal velocity)
   (c) Another model for air resistance (good for larger objects) is: "The force due to air resistance is proportional to the square of velocity." In this case, if we let downard motion be positive velocity, then we get:
   $m\dfrac{dv}{dt} = mg - kv^2$ if the object is moving downward (most common use)
   $m\dfrac{dv}{dt} = mg + kv^2$ if the object is moving upward.
   (d) *Comment*: Terminal velocity is the equilibrium velocity in an air resistance model (it is the velocity that you approach at time increases).

4. *Object in a liquid*: Much is the same in a liquid, except there is a bouyant force. The resistance is often modeled as a constant times velocity (same as with air resistance). The bouyant force is the weight of the liquid displaced by the object (let b be the mass of the liquid displaced by the object). In any case, you get a differential equation that looks something like $m\dfrac{dv}{dt} = mg - kv - bg$

5. *Changing gravity,* i.e. *objects far away from the Earth's surface*: Let x be the distance from the surface of the earth and R be the radius of the Earth. In terms of x, the force due to gravity is given by $F = -\frac{mgR^2}{(R+x)^2}$ (law of gravitation). For small values of x (near the Earth's surface) notice that $-\frac{mgR^2}{(R+x)^2} \approx -mg$. Thus, we get $m\dfrac{dv}{dt} = -\dfrac{mgR^2}{(R+x)^2}$. Note: x, v, and t depend on each other, we need to simplify/rewrite this equation in terms of only two variables:
   By definition, note that $v = \frac{dx}{dt}$. And the chain rule gives $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$. With this, we can rewrite the differential equation in terms of v and x to look like $mv\frac{dv}{dx} = -\frac{mgR^2}{(R+x)^2}$.
   The book shows how to find *escape* velocity which is the intial velocity needed to completely escape the Earth's pull due to gravity.

RLC Circuit Models

Consider the flow of electricity through a series circuit containing a resistor, an inductor, and a capacitor (called an RLC circuit). The total charge on the capacitor at time t is $q = q(t)$ in coulombs (C). We also define $I = I(t) = q'(t)$ to be the current in the circuit at time t in amperes (A). Typically, our goal will be to find the function $q(t)$.
First, let me define some constants, variables and units.

Definitions and Kirchoff's circuit laws:

- Kirchoff's second law states: *In a closed circuit the impressed voltage is equal to the sum of the voltage drops in the rest of the circuit*
- We will let $E = E(t)$ be the impressed voltage in volts (V), which is the incoming voltage to the circuit.
- Laws of electricity:
  1. The voltage drop across the resistor is proportional to the current that flows through it.
     We write $RI = Rq'$, where R is the proportionality constant due to resistance.
     We call R the resistance with the unit ohms (Ω).
  2. The voltage drop across the capacitor is proportional to the total charge on the capacitor.
     Convention is to write $\frac{1}{C}q$, where $\frac{1}{C}$ is proportionality constant due to the capacitor.
     We call C the capacitance with the unit farads (F).
  3. The voltage drop across the inductor is proportional to the derivative of the current.
     We write $LI' = Lq''$, where L is the proportionality constant due to the inductor.
     We call L the inductance with the unit henrys (H).
- The units are related as follows: $V = \Omega \cdot A = \frac{C}{F}$, and $\Omega = \frac{H}{s}$

Putting these laws together, we have

$$Lq'' + Rq' + \frac{1}{C}q = E(t)$$

Examples:

1. A series circuit has a capacitor of 0.00003 F, a resistor of 200 Ω, and an inductor of 0.6 H. There is no impressed voltage. The initial charge on the capacitor is 0.0001 C and there is no initial current. Set up the differential equation and initial conditions for the charge $q(t)$.

   Solution: You are given $C = 0.00003$, $R = 200$, $L = 0.6$, and $E(t) = 0$.
   Therefore, $0.6q'' + 200q' + (1/0.00003)\, q = 0$, with $q(0) = 0.0001$ and $q'(0) = 0$.

2. A series circuit has a capacitor of 0.0002 F and an inductor of 1.5 H (and no resistor). There is no impressed voltage. The initial charge on the capacitor is 0.005 C and there is no initial current. Set up the differential equation and initial conditions for the charge $q(t)$.

   Solution:You are given $C = 0.0002$, $R = 0$, $L = 1.5$, and $E(t) = 0$.
   Therefore, $1.5q'' + (1/0.0002)\, q = 0$, with $q(0) = 0.005$ and $q'(0) = 0$.