3.3: Homogeneous Constant Coefficient 2nd Order (Complex Roots)

Before I discuss the motivation of this method, let me give away the 'punchline'. In other words, let me show how easy it is to solve these problems once you know the general result, then we'll discuss the theoretical underpinnings:

Solutions for the Complex Root Case:
If $ar^2 + br + c = 0$ has complex roots $r = \lambda \pm \omega i$, then the general solution to $ay'' + by' + cy = 0$ is given by

$$y(t) = e^{\lambda t} \left(c_1 \cos(\omega t) + c_2 \sin(\omega t) \right).$$

Examples:

1. Give the general solution to $y'' + 3y' + \frac{10}{4} y = 0$.

   Solution: The equation $r^2 + 3r + \frac{10}{4} = 0$ has roots $r = \frac{-3 \pm \sqrt{9-10}}{2} = -\frac{3}{2} \pm \frac{1}{2} i = \lambda \pm \omega i$.
   The general solution is $y = e^{-\frac{3}{2}t} \left(c_1 \cos\left(\frac{1}{2}t\right) + c_2 \sin\left(\frac{1}{2}t\right) \right)$.

2. Give the general solution to $y'' - 4y' + 6y = 0$.

   Solution: The equation $r^2 - 4r + 6 = 0$ has roots $r = \frac{4 \pm \sqrt{-8}}{2} = 2 \pm \sqrt{2} i = \lambda \pm \omega i$.
   The general solution is $y = e^{2t} \left(c_1 \cos\left(\sqrt{2}t\right) + c_2 \sin\left(\sqrt{2}t\right) \right)$.

Examples with initial conditions:

1. Solve $y'' + 25y = 0$ with $y(0) = 2$ and $y'(0) = 3$.

   Solution: The equation $r^2 + 25 = 0$ has roots $r_1 = 0 \pm 5i = \lambda \pm \omega i$.
   The general solution is $y = c_1 \cos(5t) + c_2 \sin(5t)$.
   Note that $y' = -5c_1 \sin(5t) + 5c_2 \cos(5t)$.

   $$\begin{aligned} y(0) = 2 \quad &\Rightarrow \quad c_1 + 0 = 2 \Rightarrow c_1 = 2 \\ y'(0) = 3 \quad &\Rightarrow \quad 0 + 5c_2 = 3 \Rightarrow c_2 = \tfrac{3}{5} \end{aligned}$$

   Thus, the solution is $y(t) = 2\cos(5t) + \frac{3}{5}\sin(5t)$.

2. Solve $y'' - 4y' + \frac{25}{4} y = 0$ with $y(0) = -1$ and $y'(0) = 4$.

   Solution: The equation $r^2 - 4r + \frac{25}{4} = 0$ has roots $r = \frac{4 \pm \sqrt{-9}}{2} = 2 \pm \frac{3}{2} i$.
   The general solution is $y = e^{2t} \left(c_1 \cos\left(\frac{3}{2}t\right) + c_2 \sin\left(\frac{3}{2}t\right) \right)$
   Note that $y' = 2e^{2t} \left(c_1 \cos\left(\frac{3}{2}t\right) + c_2 \sin\left(\frac{3}{2}t\right) \right) + e^{2t} \left(-\frac{3}{2} c_1 \sin\left(\frac{3}{2}t\right) + \frac{3}{2} c_2 \cos\left(\frac{3}{2}t\right) \right)$.
   Substituting in the initial conditions gives

   $$\begin{aligned} y(0) = -1 \quad &\Rightarrow \quad c_1 + 0 = -1 \Rightarrow c_1 = -1 \\ y'(0) = 4 \quad &\Rightarrow \quad 2(c_1 + 0) + \left(0 + \tfrac{3}{2} c_2\right) = 4 \Rightarrow \tfrac{3}{2} c_2 = 6 \Rightarrow c_2 = 4 \end{aligned}$$

   Thus, the solution is $y(t) = e^{2t} \left(-\cos\left(\frac{3}{2}t\right) + 4\sin\left(\frac{3}{2}t\right) \right)$.

Some Observations and Experiments:

1. Consider $y'' + y = 0$. By guess and check, we can see that $y_1(t) = \cos(t)$ and $y_2(t) = \sin(t)$ are two solutions. You can verify this by taking derivatives. From what we discussed in section 3.2, we know that $y(t) = c_1 \cos(t) + c_2 \sin(t)$ is the general solution (notice that the Wronskian is never zero).
   Now compare this to the characteristic equation: $r^2 + 1 = 0$ has roots $r_1 = -i$ and $r_2 = i$. In this case, $\lambda = 0$ and $\omega = 1$. So we see in this example that there seems to be some connection between complex roots and solutions that involve Sine and Cosine.

2. Let's explore more: Consider $y'' + 9y = 0$. Again by guess and check, notice that $y_1(t) = \cos(3t)$ and $y_2(t) = \sin(3t)$ are solutions. Thus, the general solution is $y(t) = c_1 \cos(3t) + c_2 \sin(3t)$.
   Comparing the characteristic equation: $r^2 + 9 = 0$ has roots $r = \pm 3i$. In this case, $\lambda = 0$ and $\omega = 3$. Notice the connection between the number 3 and the coefficients inside the trig functions.

3. Now consider $y'' + 2y' + 17y = 0$. Guess and check is harder here, so let's go straight to the characteristic equation: $r^2 + 2r + 17 = 0$ has roots $r = \frac{-2 \pm \sqrt{4-68}}{2} = -1 \pm 4i$. Based on what we saw in the last two examples, we might guess that our solutions will involve $\cos(4t)$ and $\sin(4t)$. If we treat the real part of the root the same way we treat real roots, then we also might guess that our solutions will involve e^{-t}. You can check that $y_1(t) = e^{-t} \cos(4t)$ and $y_2(t) = e^{-t} \sin(4t)$ are indeed solutions (compute y' and y'') and you can check that the Wronskian is not zero.

4. See the next page for a derivation that isn't guess and check.

Euler's Formula and Derivation of the Solution

1. In section 3.1 (for real roots), we wrote all our solutions as combinations of $e^{r_1 t}$ and $e^{r_2 t}$. From our observations on the previous page, it would be nice to define $e^{\omega i}$ so that it somehow gave answers involving Cosines and Sines. In addition, using Taylor series, in my review of complex numbers (read that review sheet for more details), we saw that the following expressions are the same

$$e^{\omega i} = \cos(\omega t) + i\sin(\omega t).$$

   This is all coming together nicely. We will use this definition and it will give answers in the form we are seeing in our examples!

2. If you start with $ay'' + by' + cy = 0$ and get a characteristic equation $ar^2 + br + c = 0$ that has the complex roots $r_1 = \lambda + \omega i$ and $r_2 = \lambda - \omega i$, then, using the same method from 3.1 along with Euler's formula, you get the following:

$$
\begin{aligned}
y(t) &= a_1 e^{r_1 t} + a_2 e^{r_2 t} = a_1 e^{\lambda t + \omega t i} + a_2 e^{\lambda t - \omega t i} && (1)\\
&= a_1 e^{\lambda t} e^{\omega t i} + a_2 e^{\lambda t} e^{-\omega t i} = e^{\lambda t}\left(a_1 e^{\omega t i} + a_2 e^{-\omega t i}\right) && (2)\\
&= e^{\lambda t}\left(a_1\cos(\omega t) + a_1 i\sin(\omega t) + a_2\cos(-\omega t) + a_2 i\sin(-\omega t)\right) && (3)\\
&= e^{\lambda t}\left(a_1\cos(\omega t) + a_1 i\sin(\omega t) + a_2\cos(\omega t) - a_2 i\sin(\omega t)\right) && (4)\\
&= e^{\lambda t}\left((a_1 + a_2)\cos(\omega t) + (a_1 i - a_2 i)\sin(\omega t)\right) && (5)\\
&= e^{\lambda t}\left(c_1\cos(\omega t) + c_2\sin(\omega t)\right) && (6)
\end{aligned}
$$

Note: In going from lines (3) to (4), we use the fact that $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$ which are well known facts that always hold for these functions. These identities say that $\cos(x)$ is symmetric about the y-axis (*i.e.* it is an 'even' function) and that $\sin(x)$ gives the same graph if you reflect across the y-axis, then reflect across the x-axis (*i.e.* it is an 'odd' function).

Also note that in line (6), we are writing $c_1 = a_1 + a_2$ and $c_2 = a_1 i - a_2 i$. In this course, we will only give initial conditions that involve real numbers, so c_1 and c_2 will always be real numbers, even if you left the i in the general answer (which is fine if you do that), when you plug in the initial conditions and solve you would also find that the numbers in front of $\cos(\omega t)$ and $\sin(\omega t)$ are always real numbers in this class. (Ask me about this in office hours and I can show you what I mean).