## 3.8: Analysis from the Perspective of Beats

If the forced undamped oscillator $mu'' + ku = F_0 \cos(\omega t)$ is started from rest ($u(0) = u'(0) = 0$), then the coefficients of the homogenous solution are particularly nice.

*Recall from my 3.8 review*: If $\omega \neq \omega_0$, then a particular solution has the form $U(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$ and the general solution is:

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

Using $u'(0) = 0$ implies $c_2 = 0$ and $u(0) = 0$ implies $c_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)}$. Thus,

$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}(-\cos(\omega_0 t) + \cos(\omega t)).$$

Some useful trig identities ('superposition of trig functions as products'):

$$\cos(x) - \cos(y) = 2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{y-x}{2}\right) \quad \text{and} \quad \sin(x) - \sin(y) = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Using the first one with $x = \omega t$ and $y = \omega_0 t$ gives

$$-\cos(\omega_0 t) + \cos(\omega t) = 2\sin\left(\frac{\omega + \omega_0}{2} t\right)\sin\left(\frac{\omega - \omega_0}{2} t\right).$$

So the general solution to the starting at rest case can be written as:

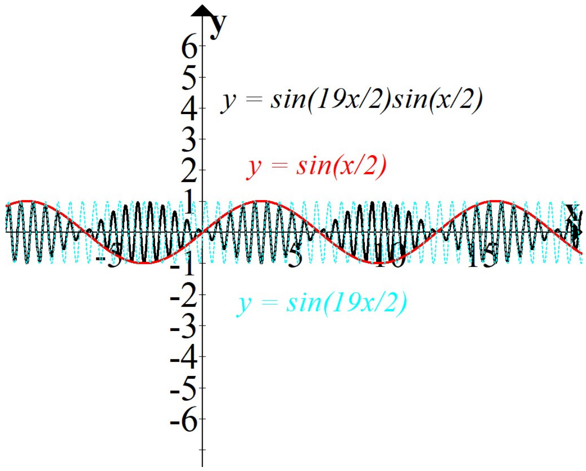
$$u(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{\omega + \omega_0}{2} t\right)\sin\left(\frac{\omega - \omega_0}{2} t\right).$$

Note that $\frac{\omega + \omega_0}{2}$ is the average frequency and $\frac{\omega - \omega_0}{2}$ is half the difference between the frequencies. You will see in the examples on the next page that these are useful numbers if you are trying to graph these functions.

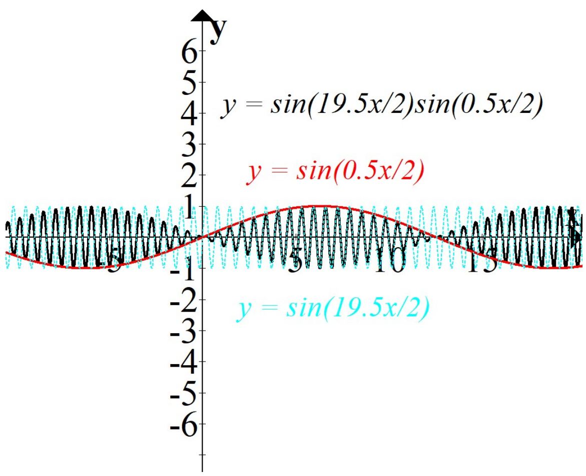
**Visualizing Answers involving two waves**

Examples:

1.
$$\frac{1}{2}(\cos(9t) - \cos(10t)) = \sin\left(\frac{10+9}{2}t\right)\sin\left(\frac{10-9}{2}t\right) = \sin\left(\frac{19}{2}t\right)\sin\left(\frac{1}{2}t\right)$$




2.
$$\frac{1}{2}(\cos(9.5t) - \cos(10t)) = \sin\left(\frac{10+9.5}{2}t\right)\sin\left(\frac{10-9.5}{2}t\right) = \sin\left(\frac{19.5}{2}t\right)\sin\left(\frac{0.5}{2}t\right)$$




Comments and conclusions:

For $mu'' + ku = F_0\cos(\omega t)$ with $u(0) = u'(0) = 0$, the solution will exhibit beats and it will look like:

$$u(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)}\sin\left(\frac{\omega+\omega_0}{2}\,t\right)\sin\left(\frac{\omega-\omega_0}{2}t\right).$$

Note that as $\omega \to \omega_0$, the size of amplitude of this will increase because $\frac{2F_0}{m(\omega_0^2-\omega^2)}$ will get larger and larger. When $\omega = \omega_0$, we call this resonance and you can see the solution for this situation in my other review sheet on 3.8.