6.1: Definition of the Laplace Transform

For a given function $f(t)$, we define the **Laplace transform**, $\mathcal{L}f(t)$, by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt.$$

This function is defined, on some domain for s, if $f(t)$ is piecewise continuous and eventually at most exponential.

Several basic examples:

1. For the constant function $f(t) = 1$, we get

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \lim_{A\to\infty} \left.\frac{-e^{-st}}{s}\right|_0^A = \frac{1}{s}$$

2. For the function $f(t) = t$, we get (use by-parts)

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} t\,dt = \lim_{A\to\infty} \left.\frac{-te^{-st}}{s} - \frac{e^{-st}}{s^2}\right|_0^A = \frac{1}{s^2}$$

3. For the function $f(t) = e^{at}$, we get

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-(s-a)t}\,dt = \lim_{A\to\infty} \left.\frac{-e^{-(s-a)t}}{s-a}\right|_0^A = \frac{1}{s-a}$$

4. For the piecewise function $f(t) = \begin{cases} 3, & 0 \le t \le 2; \\ 0, & t > 2. \end{cases}$, we get

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^2 3e^{-st}\,dt = \left.\frac{-3e^{-st}}{s}\right|_0^2 = \frac{-3e^{-2s}+3}{s}$$

5. Using integration by parts (twice) and rearranging you can find

$$\mathcal{L}\{\sin bt\} = \int_0^\infty e^{-st}\sin(bt)\,dt = \frac{b}{s^2+b^2}$$

6. Note the general fact

$$\mathcal{L}\{e^{at}f(t)\}(s) = \int_0^\infty e^{-(s-a)t} f(t)\,dt = \mathcal{L}\{f(t)\}(s-a)$$

   You can use this for a problem like:

$$\mathcal{L}\{e^{at}\sin(bt)\}(s) = \mathcal{L}\{\sin(bt)\}(s-a) = \frac{b}{(s-a)^2+b^2}$$

For a list of all the essential relationships, see the Laplace transform fact sheet and transform table.
An important fact: The Laplace transform is **linear** which means

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}.$$

For example, if you need the Laplace transform of $3 + 2t + 4\sin(5t)$, then you can write

$$\mathcal{L}\{3 + 2t + 4\sin(5t)\} = 3\mathcal{L}\{1\} + 2\mathcal{L}\{t\} + 4\mathcal{L}\{\sin(5t)\} = \frac{3}{s} + \frac{2}{s^2} + \frac{4\cdot 5}{s^2+25}$$