6.4: Solving Discontinuous Forcing Problems

To solve $ay'' + by' + cy = g(t)$ where $g(t)$ has discontinuous steps, we can:

1. Rewrite $g(t)$ in terms of step functions (see the 6.3 review).
2. Take the Laplace transform of both sides.
   Use $\mathcal{L}\{u_c(t)f(t)\} = e^{-cs}\mathcal{L}\{f(t+c)\}$.
3. Solve for $\mathcal{L}\{y\}$. Then use partial fractions.
4. Take the inverse Laplace transform.
   Use $\mathcal{L}^{-1}\{e^{-cs}F(s)\}(t) = u_c(t)\left(\mathcal{L}^{-1}\{F(s)\}(t-c)\right)$.

Try solving these examples (Solutions follow):

1. Solve $y'' - 3y' + 2y = \begin{cases} 1 & ,0 \le t < 1; \\ 0 & ,t \ge 1. \end{cases}$ with $y(0) = 0$, $y'(0) = 0$.
2. Solve $y'' - 4y = \begin{cases} 2 & ,0 \le t < 3; \\ 5 & ,3 \le t < 10; \\ 0 & ,t \ge 10. \end{cases}$ with $y(0) = 0$, $y'(0) = 1$.
3. Solve $y'' + y = \begin{cases} 0 & ,0 \le t < 3; \\ t-3 & ,3 \le t < 5; \\ 2 & ,t \ge 5. \end{cases}$ with $y(0) = 0$, $y'(0) = 0$.

Solutions:

1. Solve $y'' - 3y' + 2y = \begin{cases} 1 & ,0 \le t < 1; \\ 0 & ,t \ge 1. \end{cases}$ with $y(0) = 0$, $y'(0) = 0$.
   (a) *Step Functions*: The forcing function can be written as $1 - u_1(t)$.
   (b) *Laplace Transform*: $\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{1\} - \mathcal{L}\{u_1(t)\}$.
   (c) *Use Rules and Solve*: $s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 3(s\mathcal{L}\{y\} - y(0)) + 2\mathcal{L}\{y\} = \frac{1}{s} - \frac{e^{-s}}{s}$,
   which becomes: $(s^2 - 3s + 2)\mathcal{L}\{y\} = (1 - e^{-s})\frac{1}{s}$.
   Solving for $\mathcal{L}\{y\}$ gives: $\mathcal{L}\{y\} = (1 - e^{-s})\frac{1}{s(s^2-3s+2)}$.
   (d) *Partial Fractions*:
   $\frac{1}{s(s^2-3s+2)} = \frac{1}{s(s-1)(s-2))} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$ and you find $A = \frac{1}{2}$, $B = -1$, and $C = \frac{1}{2}$.
   Thus, $\mathcal{L}\{y\} = (1 - e^{-s})\left(\frac{1/2}{s} - \frac{1}{s-1} + \frac{1/2}{s-2}\right) = \frac{1/2}{s} - \frac{1}{s-1} + \frac{1/2}{s-2} - e^{-s}\left(\frac{1/2}{s} - \frac{1}{s-1} + \frac{1/2}{s-2}\right)$.
   (e) *Inverse Laplace transform*:
   The solution is: $y(t) = \mathcal{L}^{-1}\{\frac{1/2}{s} - \frac{1}{s-1} + \frac{1/2}{s-2}\} + \mathcal{L}^{-1}\{e^{-5s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)\}$, so
   $y(t) = \frac{1}{2} - e^t + \frac{1}{2}e^{2t} + u_5(t)\mathcal{L}^{-1}\{\frac{1/2}{s} - \frac{1}{s-1} + \frac{1/2}{s-2}\}(t-5)$, so
   $y(t) = \frac{1}{2} - e^t + \frac{1}{2}e^{2t} + u_5(t)\left(\frac{1}{2} - e^{t-5} + \frac{1}{2}e^{2(t-5)}\right)$.

2. Solve $y'' - 4y = \begin{cases} 2 & ,0 \le t < 3; \\ 5 & ,3 \le t < 10; \\ 0 & ,t \ge 10. \end{cases}$ with $y(0) = 0$, $y'(0) = 1$.

(a) *Step Functions*: The forcing function can be written as $2 + 3u_3(t) - 5u_{10}(t)$.

(b) *Laplace Transform*: $\mathcal{L}\{y''\} - 4\mathcal{L}\{y\} = \mathcal{L}\{2\} + 3\mathcal{L}\{u_3(t)\} - 5\mathcal{L}\{u_{10}(t)\}$.

(c) *Use Rules and Solve*: $s^2\mathcal{L}\{y\} - 1 - 4\mathcal{L}\{y\} = \frac{2}{s} + \frac{3e^{-3s}}{s} - \frac{5e^{-10s}}{s}$,
which becomes: $(s^2 - 4)\mathcal{L}\{y\} = \frac{2}{s} + \frac{3e^{-3s}}{s} + \frac{5e^{-10s}}{s}$.
Solving for $\mathcal{L}\{y\}$ gives: $\mathcal{L}\{y\} = \frac{2}{s(s^2-4)} + \frac{3e^{-3s}}{s(s^2-4)} + \frac{5e^{-10s}}{s(s^2-4)}$.

(d) *Partial Fractions*:
$\frac{1}{s(s^2-4)} = \frac{1}{s(s-2)(s+2))} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+2}$ and you find $A = -\frac{1}{4}$, $B = \frac{1}{8}$, and $C = -\frac{1}{8}$.
Thus, $\mathcal{L}\{y\} = 2\left(\frac{-1/4}{s} + \frac{1/8}{s-2} + \frac{-1/8}{s+2}\right) + 3e^{-3s}\left(\frac{-1/4}{s} + \frac{1/8}{s-2} + \frac{-1/8}{s+2}\right) + 5e^{-10s}\left(\frac{-1/4}{s} + \frac{1/8}{s-2} + \frac{-1/8}{s+2}\right)$.

(e) *Inverse Laplace transform*:
The solution is:
$y(t) = -\frac{1}{2} + \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t}$
. $\quad +3u_3(t)\left(\frac{-1}{4} + \frac{1}{8}e^{2(t-3)} - \frac{1}{8}e^{-2(t-3)}\right)$
. $\quad +5u_{10}(t)\left(\frac{-1}{4} + \frac{1}{8}e^{2(t-10)} - \frac{1}{8}e^{-2(t-10)}\right)$.

3. Solve $y'' + y = \begin{cases} 0 & ,0 \le t < 3; \\ t-3 & ,3 \le t < 5; \\ 2 & ,t \ge 5. \end{cases}$ with $y(0) = 0$, $y'(0) = 0$.

(a) *Step Functions*:
The forcing function can be written: $u_3(t)(t-3) + u_5(t)(2-(t-3)) = u_3(t)(t-3) + u_5(t)(5-t)$.

(b) *Laplace Transform*:
$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{u_3(t)(t-3)\} + \mathcal{L}\{u_5(t)(5-t)\} = e^{-3s}\mathcal{L}\{t\} + e^{-5s}\mathcal{L}\{5-(t+5)\}$.

(c) *Use Rules and Solve*:
$s^2\mathcal{L}\{y\} - s(0) - (0) + \mathcal{L}\{y\} = e^{-3s}\frac{1}{s^2} - e^{-5s}\left(\frac{1}{s^2}\right)$,
which becomes: $(s^2+1)\mathcal{L}\{y\} = e^{-3s}\frac{1}{s^2} - e^{-5s}\left(\frac{1}{s^2}\right)$.
Solving for $\mathcal{L}\{y\}$ gives:
$\mathcal{L}\{y\} = e^{-3s}\frac{1}{s^2(s^2+1)} - e^{-5s}\left(\frac{1}{s^2(s^2+1)}\right) = e^{-3s}G(s) - e^{-5s}G(s)$.

(d) *Partial Fractions*:
$\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$ and you find $A = 0$, $B = 1$, $C = 0$, and $D = -1$.
Thus, $G(s) = \frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$.

(e) *Inverse Laplace transform*:
Since $\mathcal{L}^{-1}\{G(s)\} = t - \sin(t)$, we have:
$\mathcal{L}^{-1}\{e^{-3s}G(s)\} = u_3(t)((t-3) - \sin(t-3)$.
$\mathcal{L}^{-1}\{e^{-5s}G(s)\} = u_5(t)((t-5) - \sin(t-5))$.
The solution is:
$y(t) = u_3(t)\left(t - 3 - \sin(t-3)\right) - u_5(t)(t - 5 - \sin(t-5))$.