6.5/6.6: The Transfer Function and the Dirac Delta Function

Terminology: Consider a forced mass-spring system: $my'' + \gamma y' + ky = f(t)$, $y(0) = 0$, $y'(0) = 0$. Taking the LaPlace Transform of both sides (and using the initial conditions) gives $(ms^2 + \gamma s + k)\mathcal{L}\{y(t)\} = \mathcal{L}\{f(t)\}$. Thus,

$$\mathcal{L}\{y(t)\} = \frac{1}{ms^2 + \gamma s + k}\mathcal{L}\{f(t)\} \quad \Longleftrightarrow \quad Y(S) = G(S)F(S)$$

In engineering and signal processing, we say:

- $F(s) = \mathcal{L}\{f(t)\}$ is the *input* which is the LaPlace Transform of the forcing function, $f(t)$.
- $G(s) = \dfrac{1}{ms^2 + \gamma s + k}$ is the *transfer function*, and $g(t) = \mathcal{L}^{-1}\{G(s)\}$ is the *impulse response*.
- $Y(s) = \mathcal{L}\{y(t)\} = G(s)F(s)$ is the *output*, and $y(t) = \mathcal{L}^{-1}\{G(s)F(s)\}$ is the solution.

Example: Consider $y'' + 9y = \cos(t)$ with $y(0) = 0$, $y'(0) = 0$.

- $f(t) = \cos(t)$ and $F(s) = \mathcal{L}\{\cos(t)\} = \dfrac{s}{s^2 + 1}$ is the *input*.
- $G(s) = \dfrac{1}{s^2 + 9}$ is the *transfer function* and $g(t) = \mathcal{L}^{-1}\{G(s)\} = \frac{1}{3}\sin(3t)$ is the *impulse response*.
- $y(t) = \mathcal{L}^{-1}\{G(s)F(s)\} = \mathcal{L}^{-1}\{\frac{1}{s^2+1}\frac{s}{s^2+1}\}$ is the solution.

Computational Shortcut

In this special case when $y(0) = 0$ and $y'(0) = 0$, there is a way to compute the solution directly without partial fractions using an integral. The theorem below gives this shortcut.

Convolution Theorem: The solution to $my'' + \gamma y' + ky = f(t)$, $y(0) = 0$, $y'(0) = 0$ is given by

$$y(t) = \int_0^t g(t - s)f(s)\,ds,$$

which is called the convolution of g and f. (proof is given on the last page of this review, but first let us use it in our example).

Example continued... Again consider $y'' + 9y = \cos(t)$ with $y(0) = 0$, $y'(0) = 0$.

Convolution integral solution: We noted that $g(t) = \mathcal{L}^{-1}\{\frac{1}{s^2+9}\} = \frac{1}{3}\sin(3t)$ and we see $f(t) = \cos(t)$. Thus the solution is given by

$$y(t) = \int_0^t \frac{1}{3}\sin(3(t - s))\cos(s)\,ds$$

Aside: Running this through a symbolic integrator gives $y(t) = \frac{1}{2}\sin^2(t)\cos(t)$. See the next page for how you can use this to quickly write the answer for any forcing function in this equation.

Many Quick Examples with the same transfer function: For all of these assume $y(0) = 0$ and $y'(0) = 0$, and notice the left-hand side matches the last example (so we know $g(t) = \frac{1}{3}\sin(3t)$ is the impulse response for all of these).

1. The solution to $y'' + 9y = e^{2t}$ is $y(t) = \displaystyle\int_0^t \frac{1}{3}\sin(3(t-s))\, e^{2s}\, ds$

2. The solution to $y'' + 9y = t\sin(4t)$ is $y(t) = \displaystyle\int_0^t \frac{1}{3}\sin(3(t-s))\, s\sin(4s)\, ds$

3. The solution to $y'' + 9y = t^4e^{7t}$ is $y(t) = \displaystyle\int_0^t \frac{1}{3}\sin(3(t-s))\, s^4e^{7s}\, ds$

Pretty neat! We can write the answers quickly using the *impulse response*, $g(t)$, in a convolution integral with whatever forcing function we are given. Let's do another example.

Example: Consider $y'' + 4y' + 4y = f(t)$ with $y(0) = 0$, $y'(0) = 0$. Write the answer as a convolution.

Solution:
Transfer Function: $G(s) = \dfrac{1}{s^2 + 4s + 4} = \dfrac{1}{(s+2)^2}$ and
Impulse Response: $g(t) = \mathcal{L}^{-1}\{\frac{1}{(s+2)^2}\} = te^{-2t}$.
Thus, for any $f(t)$, the solution can be written as a convolution in the form

$$y(t) = \int_0^t (t-s)e^{-2(t-s)} f(s)\, ds$$

If that is too abstract for you, consider $f(t) = 25\sin(t)$, so we have $y'' + 4y' + 4y = 25\sin(t)$ with $y(0) = 0$, $y'(0) = 0$. The solution is

$$y(t) = \int_0^t (t-s)e^{-2(t-s)}\, 25\sin(s)\, ds$$

and a symbolic integrator give the solution $y(t) = 4e^{-2t} + 5te^{-2t} - 4\cos(t) + 3\sin(t)$.
(*Side note*: I hope you see the repeated root and particular solution in this answer that we would have gotten if we solved the way we learned earlier in the term!)

Go to the next pages for a discussion of what is meant by the term 'impulse response'

The Dirac Delta Function and The Impulse Response:

A natural question is: Does $g(t) = \mathcal{L}^{-1}\{\frac{1}{ms^2+\gamma s+k}\}$ represent a solution to a differential equation?

If so, that would mean that $\mathcal{L}\{y\} = G(s)$. But in general we saw that the LaPlace Transform of $my'' + \gamma y' + ky = f(t)$ with $y(0) = 0$, $y'(0) = 0$ is

$$Y(s) = G(s)F(s) \iff \mathcal{L}\{y(t)\} = G(s)\mathcal{L}\{f(t)\}$$

So the only way $Y(s) = G(s)$ is if $F(s) = 1$. That is, only if the input is $F(s) = \mathcal{L}\{f(t)\} = 1$.
We haven't encounter a function yet where the LaPlace Transform is 1. In fact, all our LaPlace Transforms give us functions in terms of the variable 's'. We have never gotten a constant. The truth is, there is NO such function, but we can approach such a function and the thing we approach is given a name.

Approaching $\mathcal{L}\{f(t)\} = 1$
Let $h_\epsilon(t) = \begin{cases} 1/\epsilon & , t \le \epsilon; \\ 0 & , \text{otherwise.} \end{cases}$
Note: This can also be written in terms of a step function, namely, $h_\epsilon(t) = \frac{1}{\epsilon}(1 - u_\epsilon(t))$.
Also note: As ϵ gets smaller, the interval where the function not zero gets smaller but $\frac{1}{\epsilon}$ gets larger.
Computing the Laplace transform gives:

$$\mathcal{L}\{h_\epsilon(t)\} = \frac{1}{\epsilon}\frac{(1 - e^{-\epsilon s})}{s} = \frac{1 - e^{-\epsilon s}}{\epsilon s}$$

Then using L'Hopital's rule gives

$$\lim_{\epsilon \to 0} \mathcal{L}\{h_\epsilon(t)\} = \lim_{\epsilon \to 0} \frac{1 - e^{-\epsilon s}}{\epsilon s} \overset{H}{=} \lim_{\epsilon \to 0} \frac{se^{-\epsilon s}}{s} = 1$$

Thus, as $\epsilon \to 0$, $h_\epsilon(t)$ looks more and more like a function $f(t)$ that satisfies $\mathcal{L}\{f(t)\} = 1$.

The Dirac Delta Function:
Let $\delta_0(t)$ be defined such that $\mathcal{L}\{\delta_0(t)\} = 1$ and we think of $\delta_0(t)$ as what we get from the limiting process above (the limit of $h_\epsilon(t)$ at $\epsilon \to 0$). We call this the Dirac Delta function (named for the physicist/mathematician who defined and studied it in the 1930's). It is not really a function ('infinite' at $t = 0$ and 0 everywhere else doesn't make sense as a function), but we can still talk about the LaPlace Transform of this object and think about resulting solutions to the differential equation.

We can also define $\delta_c(t)$ to be an 'impulse' at time $t = c$ (instead of $t = 0$). It can be defined in a similar way to the previous page with a limiting process around the time $t = c$. In this case, $h_\epsilon(t) = \frac{1}{2\epsilon}(u_{c-\epsilon}(t) - u_{c+\epsilon}(t))$, so $\mathcal{L}\{h_\epsilon(t)\}$. For your own interest here is the calculation:

$$\begin{aligned} \mathcal{L}\{\delta_c(t)\} &= \lim_{\epsilon \to 0} \frac{1}{2\epsilon}\mathcal{L}\{u_{c-\epsilon}(t) - u_{c+\epsilon}(t)\} = \lim_{\epsilon \to 0} \frac{1}{2\epsilon}\frac{e^{-(c-\epsilon)s} - e^{-(c+\epsilon)s}}{s} = \lim_{\epsilon \to 0} \frac{e^{-cs}}{2s}\frac{(e^{\epsilon s} - e^{-\epsilon s})}{\epsilon} \\ &\overset{H}{=} \lim_{\epsilon \to 0} \frac{e^{-cs}}{2s}\frac{(se^{\epsilon s} + se^{-\epsilon s})}{1} = \frac{e^{-cs}}{2s}\frac{2s}{1} = e^{-cs} \end{aligned}$$

Thus,

$$\mathcal{L}\{\delta_c(t)\} = e^{-cs} \qquad \text{and, in particular,} \qquad \mathcal{L}\{\delta_0(t)\} = 1.$$

Impulse Response: We think of $\delta_c(t)$ as an instantaneous, very large magnitude 'impulse' forcing function ('smack the mass with a hammer at time $t = c$, then let it go'). This is why we call the resulting solution, the 'impulse response'. And if we want an 'impulse' at time $t = 0$, then we write

$$my'' + \gamma y' + ky = \delta_c(t), y(0) = 0, y'(0) = 0$$

and when we take the LaPlace Transform we get

$$(ms^2 + \gamma s + k)\mathcal{L}\{y\} = \mathcal{L}\{\delta_0 t\} = 1$$

So

$$\mathcal{L}\{y\} = \frac{1}{ms^2 + \gamma s + k} = G(s) \longleftrightarrow y(t) = \mathcal{L}^{-1}\{G(s)\} = g(t)$$

and that is why we call $g(t)$ the 'impulse response'.

Here are two examples

1. $y'' + y = \delta_2(t)$, $y(0) = 0$, $y'(0) = 0$.
   (a) *Laplace Transform*: $\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\delta_2(t)\}$.
   (b) *Use Rules and Solve*: $(s^2 + 1)\mathcal{L}\{y\} = e^{-2s}$ which becomes $\mathcal{L}\{y\} = e^{-2s}\frac{1}{s^2+1}$.
   (c) *Inverse Laplace transform*:
   The solution is: $y(t) = u_2(t)\sin(t - 2)$.

   Thus, the solution is $y(t) = \begin{cases} 0 & , t < 2; \\ \sin(t-2) & , t \geq 2; \end{cases}$
   So the solution is zero until the impulse happens at $t = 2$, then it gives the Sine wave with amplitude 2 which continues forever after the impulse (because there is no damping).

2. $y'' + 2y + 5y = \delta_3(t)$, $y(0) = 0$, $y'(0) = 0$.
   (a) *Laplace Transform*: $\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = \mathcal{L}\{\delta_3(t)\}$.
   (b) *Use Rules and Solve*: $(s^2 + 2s + 5)\mathcal{L}\{y\} = e^{-3s}$ which becomes $\mathcal{L}\{y\} = e^{-3s}\frac{1}{s^2+2s+5}$.
   Completing the square gives $\mathcal{L}\{y\} = e^{-3s}\frac{1}{s^2+2s+5} = e^{-3s}\frac{1}{(s+1)^2+4}$.
   (c) *Inverse Laplace transform*:
   The solution is: $y(t) = u_3(t)\frac{1}{2}e^{-(t-3)}\sin(2(t - 3))$.

   Thus, the solution is $y(t) = \begin{cases} 0 & , t < 3; \\ \frac{1}{2}e^{-(t-3)}\sin(2(t-3)) & , t \geq 3; \end{cases}$
   So the solution is zero until the impulse happens at $t = 3$, then it gives damped oscillations.

Proof of the Convolution Theorem

Thm: $\mathcal{L}^{-1}\{G(s)F(s)\} = y(t) = \int_0^t g(t-s)f(s)\,ds$

Proof of Convolution Theorem (not required for class):
Let $\mathcal{L}\{g(t)\} = G(s)$ and $\mathcal{L}\{f(t)\} = F(s)$.
We need to prove that $\mathcal{L}\{\int_0^t g(t-s)f(s)\,ds\} = G(s)F(s)$.

By definition (and using different integration variables to try to avoid confusion):

$$
\begin{aligned}
G(s)F(s) &= \int_0^\infty e^{-st}f(t)dt \int_0^\infty e^{-su}g(u)du \\
&= \int_0^\infty \int_0^\infty e^{-s(t+u)}f(t)g(u)dtdu
\end{aligned}
$$

let $w = t + u,$ so $dw = dt$ to get

$$
\begin{aligned}
&= \int_0^\infty \int_u^\infty e^{-sw}f(w-u)g(u)dwdu \\
&= \int_0^\infty \int_u^\infty e^{-sw}f(w-u)g(u)dwdu
\end{aligned}
$$

then reversing order using ideas from Math 126/324 gives

$$
\begin{aligned}
&= \int_0^\infty e^{-sw}\left(\int_0^w f(w-u)g(u)du\right)dw \\
&= \mathcal{L}\left\{\int_0^w f(w-u)g(u)du\right\}
\end{aligned}
$$