3.7 and 3.8: Vibrations Handout

Mass-Spring Systems:
An object is placed on a spring. If $u(t)$ is the displacement from rest, then we say

$$mu'' + \gamma u' + ku = F(t),$$

where m is the mass of the object, γ is the damping constant, k is the spring constant, and $F(t)$ is an external forcing function. In deriving the application, we learned various facts including: $mg - kL = 0$, $F_g = mg$, $F_s = -k(L+u)$, and $F_d = -\gamma u'(t)$, where $g = 9.8 \text{ m/s}^2 = 32 \text{ ft/s}^2$ and L is the distance the spring is stretched beyond natural length when it is at rest.

RLC circuits:
If R, C, and L are the resistance, capacitance and inductance in a circuit and $E(t)$ is the impressed voltage (incoming forcing function), then we have

$$LQ'' + RQ' + \frac{1}{C}Q = E(t),$$

where $Q(t)$ is the charge on the capacitor at time t.
Note: This is not a physics or electronics class. You really don't have to know hardly anything about forces or electronics to do well on this material. You just have to put in the numbers and solve second order systems. The point of this material is to expose you to some important applications of second order equations so that you have a physical relationship between what we are getting in the solutions and what we are seeing in the application.

Summary of Analysis:
Note: Here I state everything in terms of the mass-spring system, but, if you replace $m = L$, $\gamma = R$, $k = \frac{1}{C}$, and $F(t) = E(t)$, then the analysis is the same for the circuit application.

No Forcing: $F(t) = 0$.

1. $\gamma = 0 \Rightarrow$ No Damping: Solution looks like $u(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) = R\cos(\omega_0 t - \delta)$.
   - Natural frequency: $\omega_0 = \sqrt{k/m}$ radians/second.
   - Period: $T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{m/k}$ seconds/wave.
   - Amplitude: $R = \sqrt{c_1^2 + c_2^2}$.
2. $\gamma \geq 2\sqrt{mk} \Rightarrow$ No Vibrations: No imaginary roots, only negative real roots.
   $\gamma = 2\sqrt{mk} \Rightarrow$ **Critically Damped** and $\gamma > 2\sqrt{mk} \Rightarrow$ **Overdamped**
3. $0 < \gamma < 2\sqrt{mk} \Rightarrow$ **Damped Vibrations** (also called *Underdamped*):
   Solutions looks like $u(t) = e^{\lambda t}(c_1\cos(\mu t) + c_2\sin(\mu t)) = Re^{\lambda t}\cos(\mu t - \delta)$.
   - Quasi-frequency: $\mu = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}$ radians/second.
   - Quasi-period: $T = \frac{2\pi}{\mu}$ seconds/wave.
   - Amplitude: $Re^{\lambda t} = \sqrt{c_1^2 + c_2^2}e^{\lambda t}$, which goes to zero as $t \to \infty$.
     (because $\lambda = -\frac{\gamma}{2m}$ which is negative).

Forcing: $F(t) \neq 0$.
As we saw in 3.5, we need to find the homogeneous solution and a particular solution.
For mass-spring, we did a full analysis of the case when $\mathbf{mu''} + \gamma\mathbf{u'} + \mathbf{ku} = \mathbf{F(t)} = \mathbf{F_0}\cos(\omega\mathbf{t})$.

1. $\gamma = 0 \Rightarrow$ No Damping: Find homogenous solutions (see 'no forcing'). It will have a natural frequency of ω_0. The particular solution depends on ω and $\omega_0 = \sqrt{k/m}$.
   So the solution will look like $u(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + Y_p(t)$.

   - If $\omega \neq \omega_0$, then a particular solution looks like $Y_p(t) = \frac{F_0}{m(\omega_0^2-\omega^2)}\cos(\omega t)$.
     In this case, we noted that if $u(0) = 0$ and $u'(0) = 0$, then we get $c_2 = 0$ and $c_1 = -\frac{F_0}{m(\omega_0^2-\omega^2)}$, which let to $u(t) = \frac{F_0}{m(\omega_0^2-\omega^2)}(-\cos(\omega_0 t) + \cos(\omega t))$, which we used an identity to write as $u(t) = \frac{2F_0}{m(\omega_0^2-\omega^2)}\sin\left(\frac{\omega+\omega_0}{2}t\right)\sin\left(\frac{\omega_0-\omega}{2}t\right)$ which we could graph by plotting a sine wave with frequency $\frac{\omega_0-\omega}{2}$ as an upper and lower bound for our "beats" amplitudes, then plot a faster wave inside with period $\frac{\omega+\omega_0}{2}$.
     (*see pictures in my lecture notes on beats*)
   - If $\omega = \omega_0$, then a particular solution looks like $Y_p(t) = \frac{F_0}{2m\omega_0}t\sin(\omega t)$. (Resonance!)
     We discussed this two ways: (1) By relying on the method of undetermined coefficients and the fact that our 'guess' gets multiplied by t when it matches the homogeneous solutions, And (2) We took the limit of the previous answer as $\omega \to \omega_0$. Both led to this answer.

2. $\gamma > 0 \Rightarrow$ Damping: Find homogenous solutions (see 'no forcing').
   If $\gamma < 2\sqrt{mk}$, then label μ as the quasi-frequency. If $\gamma < 2\sqrt{mk}$, then the solution will always look like:

   $$u(t) = u_c(t) + U(t) = c_1e^{\lambda t}\cos(\mu\, t) + c_2e^{\lambda t}\sin(\mu\, t) + A\cos(\omega t) + B\sin(\omega t).$$

   In all cases where $\gamma > 0$ the homogeneous solution, $u_c(t)$, goes to zero as $t \to 0$. We say the homogeneous solution, $u_c(t)$, is the **transient solution** and the particular solution, $Y_p(t)$, is the **steady state solution** (or forced response).

   - With some considerable algebra, you can get general messy formulas for A and B (see book or review sheet).
   - Amplitude of Steady State solution: $R = \sqrt{A^2+B^2} = \frac{F_0}{\sqrt{(k-m\omega^2)^2+\gamma^2\omega^2}}$.
     This depends on ω.
     Applitude is maximized when $\omega = \omega_{max} = \omega_0\sqrt{1 - \frac{\gamma^2}{2mk}} \approx \omega_0$. (if γ is close to zero)
     At this value of ω, you get $R = R_{max} = \frac{F_0}{\gamma\omega_0}\frac{1}{\sqrt{1-\frac{\gamma^2}{4mk}}}$.
     So if γ is close to zero, then the maximum amplitude of the steady state response occurs when ω is close to ω_0 (Resonance).