MATH 307D
Midterm 1
July 12, 2013

Name ______________________________ Student ID #______________

- Your exam should consist of this cover sheet, followed by 5 problems. Check that you have a complete exam.
- Unless otherwise indicated, show all your work and justify your answers.
- Unless otherwise indicated, your answers should be exact values rather than decimal approximations. For example, $\frac{\pi}{4}$ is an exact answer and is preferable to 0.7854.
- You may use a scientific calculator and one double-sided 8.5×11-inch sheet of handwritten notes. All other electronic devices, including graphing or programmable calculators, and calculators which can do calculus, are forbidden.
- The use of headphones, earbuds during the exam is not permitted. Turn off all your electronic devices and put them away.
- If you need more space, write on the back and indicate this. If you still need more space, raise your hand and I'll give you some extra paper to staple onto the back of your test.
- Academic misconduct will guarantee a score of zero on this exam. DO NOT CHEAT.

Problem	Points	S C O R E
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 points) Find the general solution of the differential equation

$$y' = (\sin t)(y-2)^2.$$

Solution: Separate variables:

$$\frac{dy}{(y-2)^2} = \sin t\,dt \;\Rightarrow\; \int \frac{dy}{(y-2)^2} = \int \sin t\,dt \;\Rightarrow\; -\frac{1}{y-2} = -\cos t + C$$

Solving for y, we get:

$$y = 2 + \frac{1}{\cos t - C}.$$

Here, C can take any real values. We also lost the solution $y = 2$. It cannot be included into the formula above, with C. The general solution of this differential equation:

$$\boxed{y(t) = 2 + \frac{1}{\cos t - C},\quad C \text{ is any real number, and }\; y(t) = 2}$$

2. (10 points) Solve the initial value problem:

$$y' = 2y + e^t + e^{2t}, \quad y(0) = 0.$$

Solution: This is a linear equation. First, let us solve the corresponding homogeneous equation:

$$y' = 2y \Rightarrow y = Ce^{2t}.$$

Then, let us solve the inhomogeneous equation using the method of variation of parameters. Assume $C = C(t)$ is a function of t, and plug it into the original equation: the left-hand side equals

$$y'(t) = (C(t)e^{2t})' = C'(t)e^{2t} + C(t)2e^{2t},$$

and the right-hand side equals

$$2C(t)e^{2t} + e^t + e^{2t}.$$

Therefore,

$$C'(t)e^{2t} = e^t + e^{2t} \Rightarrow C'(t) = e^{-t} + 1 \Rightarrow C(t) = -e^{-t} + t + K.$$

Plug this into $y(t)$ and finish the solution:

$$y(t) = \left[-e^{-t} + t + K\right] e^{2t} = -e^t + te^{2t} + Ke^{2t}.$$

This is the general solution. Plug in $t = 0$ and solve for K: $-1 + 0 + K = 0 \Rightarrow K = 1$, so the answer is

$$\boxed{y(t) = -e^t + te^{2t} + e^{2t}}$$

3. (10 points) Consider the equation

$$y' = \frac{-y}{(y^2+1)^3}.$$

Analyze it qualitatively. Find constant solutions and classify them (stable/unstable/semistable). Draw the graph of increasing/decreasing/constant solutions.

Solution: The right-hand side (denote it by $f(y)$) satisfies:

$$f(y) = \begin{cases} < 0, & \text{if } y > 0; \\ = 0, & \text{if } y = 0; \\ > 0, & \text{if } y < 0 \end{cases}$$

So the solutions are increasing below zero and decreasing above zero. The zero solution, $y(t) = 0$, is stable.

4. (10 points) Find $y(1), y(2), y(3)$ using Euler's method with $h = 1$ for the following initial value problem:

$$y' = y^2 - yt,\ y(0) = 1.$$

Solution:

- $y'(0) = y^2(0) - y(0) \cdot 0 = 1;$
- $y(h) = y'(0)h + y(0) = 2,\ y'(h) = y^2(h) - y(h)h = 2^2 - 2 \cdot 1 = 2;$
- $y(2h) = y'(h)h + y(h) = 4,\ y'(2h) = y^2(2h) - y(2h)(2h) = 16 - 4 \cdot 2 = 8;$
- $y(3h) = y'(2h)h + y(2h) = 12.$

So

$$\boxed{y(1) = 2,\ y(2) = 4,\ y(3) = 12}$$

5. (10 points) You have one million dollar (lottery jackpot) in your savings account. The annual interest is 5%, but it is added continuously. You continuously withdraw one hundred thousand dollars annually to sustain your living. How long can you live on this money?

Solution: Assume for simplicity $100,000\$ = 1$. Let $y(t)$ be the amount of money in your account at time t. Then

$$y'(t) = 0.05y(t) - 1, \quad y(0) = 10.$$

Let us solve this differential equation by the method of variation of parameters. First, let us solve the homogeneous equation:

$$y'(t) = 0.05y(t) \quad \Rightarrow \quad y(t) = Ce^{t/20}.$$

Then let $C = C(t)$ depend on t and plug into the inhomogeneous (original) equation:

$$C'(t)e^{t/20} + C(t)\frac{1}{20}e^{t/20} = \frac{1}{20}C(t)e^{t/20} - 1 \quad \Rightarrow \quad C'(t) = -e^{-t/20} \quad \Rightarrow \quad C(t) = 20e^{-t/20} + K.$$

PLug it back into $y(t) = C(t)e^{t/20}$ and get:

$$y(t) = \left(20e^{-t/20} + K\right)e^{t/20} = 20 + Ke^{t/20}.$$

Now, let us solve the initial value problem:

$$y(0) = 10 \quad \Rightarrow \quad 20 + K = 10 \quad \Rightarrow \quad K = -10.$$

Therefore,

$$y(t) = 20 - 10e^{t/20}.$$

Let us solve the equation $y(t) = 0$ for t: $e^{t/20} = 2 \Rightarrow t = \boxed{20\ln 2} \approx 14.$