1. (11 pts)

(a) Find the general explicit solution to $\frac{dy}{dt} + \frac{2}{t}y = t$.

$$\mu(t) = e^{\int \frac{2}{t}dt} = e^{2\ln|t|} = e^{\ln(t^2)} = t^2 \quad \cdot t^2$$

$$t^2 \frac{dy}{dt} + 2ty = t^3$$

$$\frac{d}{dt}(t^2 y) = t^3$$

$$t^2 y = \frac{1}{4}t^4 + c$$

$$\boxed{y(t) = \frac{1}{4}t^2 + \frac{c}{t^2}}$$

(b) Consider $\frac{dy}{dt} = (1 + 4e^t)(y - 3)^2$. (Put final solutions in the boxes provided)

i. Find the explicit solution with $y(0) = 4$. Solution: $\boxed{y(t) = 3 - \frac{1}{t + 4e^t - 5}}$

ii. Find the explicit solution with $y(5) = 3$. Solution: $\boxed{y(t) = 3}$

EQUILIBRIUM SOL'N $\quad y(t) = 3$

IF $y \neq 3$, THEN $\quad \int \frac{1}{(y-3)^2}\,dy = \int (1 + 4e^t)\,dt$

$$-\frac{1}{y-3} = t + 4e^t + c$$

$$y - 3 = -\frac{1}{t + 4e^t + c}$$

$$y(t) = 3 - \frac{1}{t + 4e^t + c}$$

$y(0) = 4 \Rightarrow \quad 4 = 3 - \frac{1}{0 + 4 + c} \Rightarrow 1 = -\frac{1}{4 + c}$

$$4 + c = -1$$

$$c = -5$$

2. (11 pts)

(a) Find an implicit solution to $(5xe^y + 4y^3)\dfrac{dy}{dx} = -5e^y$ with $y(1) = 0$.

$5e^y + (5xe^y + 4y^3)\frac{dy}{dx} = 0$ $\qquad$ $\frac{\partial}{\partial y}(5e^y) = 5e^y = \frac{\partial}{\partial x}(5xe^y + 4y^3)$ ✓

$\int 5e^y\,dx = 5xe^y + C_1(y)$ $\qquad$ Sol'n: $5xe^y + y^4 = C$

skip (exact equations)

$y(1) = 0 \Rightarrow 5(1)e^0 + 0^4 = C \Rightarrow C = 5$ $\qquad$ $\boxed{5xe^y + y^4 = 5}$

(b) Consider $2y\dfrac{dy}{dx} = 3e^x + y^2$ with $y(0) = 2$.
Use $u = y^2$ to transform the differential equation into one involving x, u, and $\frac{du}{dx}$.
Solve the new equation and give the explicit solution to the original initial value problem.

$$\left.\begin{aligned} u &= y^2 \\ \frac{du}{dx} &= 2y\frac{dy}{dx} \end{aligned}\right\}$$

$\frac{du}{dx} = 3e^x + u$

$\frac{du}{dx} - u = 3e^x$ $\qquad$ $\mu(x) = e^{-x}$

$e^{-x}\frac{du}{dx} - e^{-x}u = 3e^x e^{-x}$

$\frac{d}{dx}\left[e^{-x}u\right] = 3$

$e^{-x}u = 3x + C$

$u = 3xe^x + Ce^x$

$\Rightarrow\ y^2 = 3xe^x + Ce^x$

$y(x) = \pm\sqrt{3xe^x + Ce^x}$

$y(0) = 2 \Rightarrow 2 = \sqrt{0 + C} \Rightarrow C = 4$

$\boxed{y(x) = \sqrt{3xe^x + 4e^x}}$

3. (8 pts) Dr. Loveless filled his bathtub, pulled the plugged and used a stopwatch to record how long it took it to empty. Using the model from one of your homework questions and replacing the constants with measurements from the bathtub, we get the differential equation

$$\frac{dh}{dt} = -0.002\sqrt{19.6h},$$

where $h(t)$ is height in meters of the water in the bathtub after t seconds. The initial height of the water is $h(0) = 0.4$ meters.

(a) Find the explicit solution to the differential equation.

$$\int \frac{1}{\sqrt{h}}\,dh = \int -0.002\sqrt{19.6}\,dt$$

$$2\sqrt{h} = -0.002\sqrt{19.6}\,t + C$$

$$h(t) = \left(-0.001\sqrt{19.6}\,t + D\right)^2 \qquad D = \tfrac{1}{2}C$$

$$h(0) = 0.4 \Rightarrow \qquad D^2 = 0.4 \Rightarrow D = \sqrt{0.4}$$

$$\boxed{h(t) = \left(-0.001\sqrt{19.6}\,t + \sqrt{0.4}\right)^2}$$

NOTE: CHECK YOUR WORK!!!

$$\frac{dh}{dt} = 2\left(-0.001\sqrt{19.6}\right)\left(-0.001\sqrt{19.6}\,t + \sqrt{0.4}\right)$$

SAME

$$-0.002\sqrt{19.6h} = -0.002\sqrt{19.6\left(-0.001\sqrt{19.6}\,t + \sqrt{0.4}\right)^2} = -0.002\sqrt{19.6}\left(-0.001\sqrt{19.6}\,t + \sqrt{0.4}\right)$$

(b) According to this model, how long does it take for Dr. Loveless' bathtub to empty?

$$h(t) \stackrel{?}{=} 0 \Rightarrow \qquad -0.001\sqrt{19.6}\,t + \sqrt{0.4} \stackrel{?}{=} 0$$

$$\boxed{t = \frac{\sqrt{0.4}}{0.001\sqrt{19.6}} \approx 142.86 \text{ seconds}}$$

4. (8 pts) Water drips into a cone shaped hole in the ground at a constant rate and also evaporates. Using the model from homework and some estimates for constants, we get the differential equation:

$$\frac{dV}{dt} = 10 - 0.5V^{2/3},$$

where $V(t)$ is the volume of water in the puddle in cubic inches after t minutes. Assume the puddle is initially empty (so $V(0) = 0$).

(a) Use Euler's method with step size $h = 0.5$ to estimate the volume of the water in the puddle in $t = 1$ minute. Round your final answer to 3 digits after the decimal.

$t_0 = 0,\ V_0 = 0$

$\Rightarrow\ 10 - 0.5(V_0)^{2/3} = 10 - 0.5(0)^{2/3} = 10$

$t_1 = 0.5,\ V_1 = V_0 + 10h = 0 + 10(0.5) = 5$

$\Rightarrow\ 10 - 0.5(V_1)^{2/3} = 10 - 0.5(5)^{2/3} \approx 8.537991$

$t_2 = 1.0,\ V_2 = V_1 + 8.537991h = 5 + 8.537991(0.5)$

$\Rightarrow\ \boxed{V(1) \approx 9.2689956 \approx 9.269 \text{ in}^3}$

(b) Without solving, determine the value of $\lim_{t\to\infty} V(t)$. Explain your answer

EQUILIBRIUM SOL'N? $\quad 10 - 0.5V^{2/3} \stackrel{?}{=} 0$

$\Rightarrow\ V^{2/3} = 20$

VERY IMPORTANT TO NOTE THIS

$V = \pm 20^{3/2} \approx \pm 89.442719 \text{ in}^3$

IF $-20^{3/2} < V < 20^{3/2}$ THEN $\frac{dV}{dt}$ is positive (sol'ns increase)

IF $V > 20^{3/2}$ THEN $\frac{dV}{dt}$ is negative (sol'ns decrease)

IF $V < -20^{3/2}$ THEN $\frac{dV}{dt}$ is negative (sol'n decrease)

THUS, $V(t) = 20^{3/2}$ IS A STABLE EQUILIBRIUM AND $V(t) = -20^{3/2}$ IS UNSTABLE

IN PARTICULAR, IF $V(0) = 0$, THEN $\boxed{\lim_{t\to\infty} V(t) = 20^{3/2} \text{ in}^3}$

Sol'n increase and continue to increase approaching $20^{3/2}$ because $\frac{dV}{dt} > 0$

5. (12 pts) These are all short answer questions!

(a) A spherical snowball melts (loses volume) at a rate proportional to its surface area with a proportionality constant that has magnitude 0.1 in/min. Give the differential equation for the volume that only involves the variables t and V. (DO NOT SOLVE)
For reference: For a sphere, Volume $= V = \frac{4}{3}\pi r^3$, Surface Area $= S = 4\pi r^2$.

$$\frac{dV}{dt} = -0.1\,S = -0.1 \cdot 4\pi r^2 \qquad \text{Since } V = \tfrac{4}{3}\pi r^3,\ r = \left(\frac{3V}{4\pi}\right)^{1/3}$$

decreasing

ALL ACCEPTABLE

$$\boxed{\frac{dV}{dt} = -0.1 \cdot 4\pi \left(\frac{3V}{4\pi}\right)^{2/3} = -0.1\, 4\pi \left(\frac{3}{4\pi}\right)^{2/3} V^{2/3} = -0.1\,(4\pi)^{1/3}\, 3^{2/3}\, V^{2/3}}$$

(b) Historical data suggests that the population in a certain town grows at a rate proportional to the population size with proportionality constant $k = 0.03$. Due to an economic downturn that starts today, jobs become scarce and people start to leave at a constant rate of 300 people per year.
Let $y(t)$ be the number of people in the town t years after the economic downturn started.

i. Write down the differential equation that models the population.
(Big Hint: This is just like a bank account problem: Population is growing at a 3% annual rate and 300 people are 'withdrawing' from the town each year)

$$\boxed{\frac{dy}{dt} = 0.03y - 300}$$

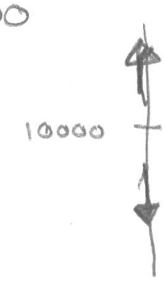
people per year

ii. Find all equilibrium solutions and classify them as stable, unstable or semistable.

$$0.03y - 300 \stackrel{?}{=} 0 \Rightarrow y = \frac{300}{0.03} = 10{,}000$$

IF $y < 10000$, THEN $\frac{dy}{dt}$ IS NEGATIVE.

IF $y > 10000$, THEN $\frac{dy}{dt}$ IS POSITIVE.



$\boxed{y(t) = 10000 \text{ IS UNSTABLE}}$

iii. Assume the town started with 4000 people. What will happen to the population size in this town as t increases? Explain your answer.

SINCE $y(0) = 4000$ IS LESS THAN 10,000, THE SOL'N WILL DECREASE WITHOUT BOUND FROM WHAT WE LEARNED IN THE PREVIOUS PART.

$\boxed{\text{EVENTUALLY EVERYONE WILL LEAVE THE TOWN}}$