Information you'll have for the final:

Table of Laplace Transforms

f	$\mathcal{L}[f]$	f	$\mathcal{L}[f]$
1	$\frac{1}{s}$	$\cos bt$	$\frac{s}{s^2+b^2}$
e^{at}	$\frac{1}{s-a}$	$\sin bt$	$\frac{b}{s^2+b^2}$
t^n	$\frac{n!}{s^{n+1}}$	$e^{at}\cos bt$	$\frac{(s-a)}{(s-a)^2+b^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$

Acceleration Due to Gravity

standard: $g = 32.2$ ft/s^2 (you can use $g = 32$)

metric: $g = 9.8$ m/s^2 (you can use $g = 10$)

1. A tank of water starts with 40 g of dye dissolved in 10 L of water. Solution containing 5 g/L of dye enters the tank at a rate of 6 L/s, mixes with the contents of the tank, and the mixture drains at a rate of 4 L/s.

Find the concentration of dye at time t. Find the limiting concentration of dye as $t \to \infty$.

Answer: the concentration is $5 - \frac{1000}{(10+2t)^3}$ *g/L. The limiting concentration is 5 g/L*

2. (a) Solve the equation

$$\frac{1}{x}y' = e^{x+y}.$$

This is a separable equation — answer: $y(x) = -\ln(-xe^x + e^x + C)$.

(b) Solve the equation

$$\frac{1}{x}y' + \frac{2}{x^2}y = \frac{e^x}{x^2}.$$

This is a 1st-order linear equation (so can use integrating factors). Answer: $y(x) = \frac{e^x}{x} - \frac{e^x}{x^2} + \frac{C}{x^2}$

3. A 2lb weight is attached to a spring, stretching it 4 feet. There is a damping force, which is equal to ~~40 lb~~ 5 lb when the weight is traveling at ~~5 ft/s~~ 20 ft/s. There's also an external force $F(t) = \frac{1}{4}\cos 3t$ lb acting on the weight.

(a) Find the quasiperiod of the system and the general solution.

(b) What is the amplitude and phase of the steady-state solution? (Your answer may involve square roots and trigonometric functions.)

The differential equation is $\frac{1}{16}u'' + \frac{1}{4}u' + \frac{1}{2}u = \frac{1}{4}\cos 3t$.

Answers: (a) Quasiperiod: $T_d = \pi$;
general solution: $u(t) = c_1 e^{-2t}\cos 2t + c_2 e^{-2t}\sin 2t - \frac{4}{145}\cos 3t + \frac{48}{145}\sin 3t$.

(b) The steady state solution is $-\frac{4}{145}\cos 3t + \frac{48}{145}\sin 3t$. *Amplitude:* $\frac{4\sqrt{145}}{145}$, *phase:* $\tan^{-1}(-12) + \pi$.

4. Match the initial value problems shown below with the graphs of their solutions:

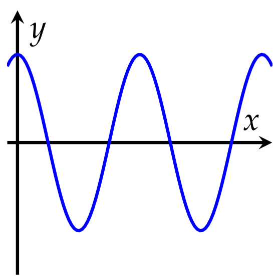
1. $\begin{cases} y'' + 5y' + 6y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$ *Answer: F.*

2. $\begin{cases} y'' - 4y' + 6y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$ *Answer: C.*

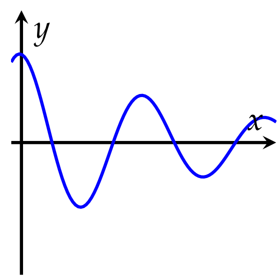
3. $\begin{cases} y'' - 5y' + 6y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$ *Answer: D.*

4. $\begin{cases} y'' + 6y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$ *Answer: A.*

5. $\begin{cases} y'' + 4y' + 6y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$ *Answer: B.*



(A)



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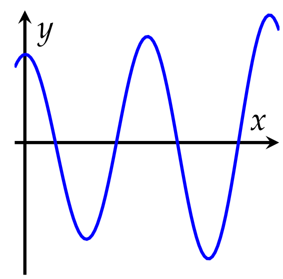

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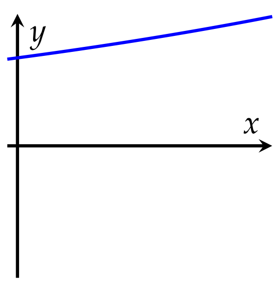

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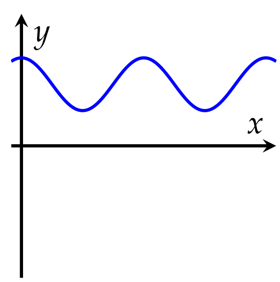

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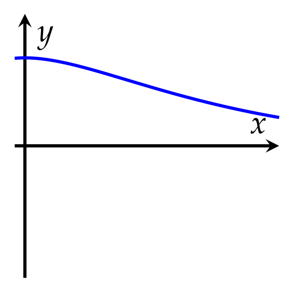

(F)

5. Solve the initial value problem

$$\begin{aligned} Q'' + 2Q' + 10Q &= E(t) \\ E(t) &= \begin{cases} -10e^{-2t}, & t < \pi \\ 0, & t \geq \pi \end{cases} \\ Q(0) &= 1 \\ Q'(0) &= -3. \end{aligned}$$

Answer: Using step functions:

$$Q(t) = 2e^{-t}\cos(3t) - e^{-t}\sin(3t) - e^{2t} + u_\pi(t)e^{-2\pi}\left[-e^{-t+\pi}\cos(3t-3\pi) + \frac{1}{3}e^{-t+\pi}\sin(3t-3\pi) + e^{-2(t-\pi)}\right]$$

In piecewise form (before simplification):

$$Q(t) = \begin{cases} 2e^{-t}\cos(3t) - e^{-t}\sin(3t) - e^{2t}, & t < \pi \\ 2e^{-t}\cos(3t) - e^{-t}\sin(3t) - e^{-2\pi}e^{-t+\pi}\cos(3t-3\pi) + \frac{1}{3}e^{-2\pi}e^{-t+\pi}\sin(3t-3\pi), & t \geq \pi. \end{cases}$$

6. Find the Laplace transform of $f(t) = t \sin t$, using the definition of the Laplace transform. You can use the facts that $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$ and $\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$.

Answer: $\mathcal{L}\{t \sin t\} = \frac{2s}{(s^2+1)^2}$.