Practice Midterm 2 Answers

1. Solve the IVP

$$y'' - 4y' + 4y = e^{2t}\cos(3t) + 4$$
$$y(0) = 0$$
$$y'(0) = 0.$$

Homogeneous solution: $y_c(t) = c_1 e^{2t} + c_2 t e^{2t}$

Particular solution format: $Y(t) = Ae^{2t}\cos(3t) + Be^{2t}\sin(3t) + C$

Particular solution: $Y(t) = -\frac{1}{9}e^{2t}\cos(3t) + 1$

General solution: $u(t) = c_1 e^{2t} + c_2 t e^{2t} - \frac{1}{9}e^{2t}\cos(3t) + 1$

Solution to IVP: $u(t) = -\frac{8}{9}e^{2t} + 2te^{2t} - \frac{1}{9}e^{2t}\cos(3t) + 1.$

2. An object weighing 96 lb is attached to a spring, stretching it 2 feet. Assume there is no damping, and that an external force $F(t) = 3\sin(4t) - \cos(4t)$ is applied to the object. At time $t = 0$, you push the object 3 feet upward from equilibrium position and give it an initial velocity of 1 ft/s downward. Find the position of the object at time t.

(Recall $g = 32\text{ft/s}^2$).

Solutions:

Equation: $3u'' + 48u = 3\sin(4t) - \cos(4t)$

Homogeneous solution: $y_c(t) = c_1\cos(4t) + c_2\sin(4t)$

Particular solution format: $Y(t) = At\cos(4t) + Bt\cos(4t)$

Particular solution: $Y(t) = -\frac{1}{8}t\cos(4t) - \frac{1}{24}t\sin(4t)$

General solution: $u(t) = -\frac{1}{8}t\cos(4t) - \frac{1}{24}t\sin(4t) + c_1\cos(4t) + c_2\sin(4t).$

Solution to IVP: $u(t) = -\frac{1}{8}t\cos(4t) - \frac{1}{24}t\sin(4t) - 3\cos(4t) + \frac{9}{32}\sin(4t).$ *(correction)*

3. A 10kg rock is attached to a spring, stretching it 2 meters.

(a) For this part only, assume there is no damping, and no external force. If at $t = 0$ the spring is stretched downward by 2m and the rock is released with initial velocity 7 m/s upward, find the period, amplitude, and phase of the motion (Your answer for the phase may involve a trigonometric function).

(b) Now assume there is damping, and that the magnitude of the damping force is 12 N when the object is traveling at 2 m/s. Find the quasi-period of the motion.

(c) How large does the damping force need to be for the system to be critically damped?

Solutions:

(a) Initial value problem: $10u'' + 49u = 0$, $u(0) = 2$, $u'(0) = -7$

Solution: $u = 2\cos(7/\sqrt{10}t) - \sqrt{10}\sin(7/\sqrt{10}t)$

Amplitude: $A = \sqrt{14}$

Phase: $\delta = \tan^{-1}(-\sqrt{10}/2)$ *(correction)*

Period: $T = 2\pi\sqrt{10}/7$

(b) ODE: $10u'' + 6u' + 49u = 0.$

Quasiperiod: $\mu = \sqrt{1924}/20 = \sqrt{481}/10.$

(c) We would need $\gamma = 14\sqrt{10}$*, which corresponds to a damping force of* $28\sqrt{10}$ *N at a velocity of 2 m/s.*

4. Given that $y_1(t) = 1/t$ is a solution to the following equation, find another solution:

$$t^2y'' + 3ty' + y = 0, \quad t > 0.$$

One possible answer: $y_2(t) = \frac{\ln t}{t}$. *The general solution is* $y(t) = \frac{C}{t} + \frac{D\ln t}{t}$.