Your Name

Student ID #

- Cellphones off please!
- You are allowed one two-sided handwritten notesheet for this midterm. You may use a scientific calculator; graphing calculators and all other course-related materials may not be used.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $e^{-5\sqrt{3}}$) unless explicity stated otherwise by the question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- There is a table of Laplace transforms and rules at the back of this exam. You may quote and use any of the formulas and rules in the table as is without having to derive them from scratch.
- This exam has 10 pages, plus this cover sheet. Please make sure that your exam is complete.
- You have 110 minutes to complete the exam.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total	80	

1. (10 total points) Find the explicit general solutions to the following differential equations. Your answer should be in the form $y = f(t)$, with an undetermined constant appearing somewhere in the equation.

   (a) (5 points)

$$t^2y' + y' = e^y$$

   (b) (5 points)

$$ty' - 2y = t^3 - t$$

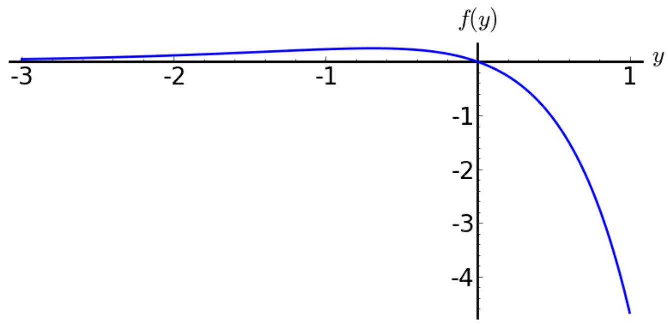
2. (10 points) Find the solution to the following initial value problem:

$$y'' + y' - 6y = 1 - e^{-t}, \qquad y(0) = 0,\ y'(0) = -1$$

3. (10 total points) Consider the autonomous differential equation

$$y' = e^y - e^{2y},$$

where y is a function of t. Below is a graph of $f(y)$ versus y:



(a) (5 points) Find all equilibrium solutions to this differential equation, and classify them according to their stability (stable, unstable or semistable). Be sure to justify your answer.

(b) (5 points) Suppose we are now looking at the solution to the above DE subject to the initial condition $y(0) = 1$. Use a single step of Euler's method to approximate the value of the solution at $t = 0.5$. You may use decimals in this part of the question, but be sure to maintain at least four digits of precision.

4. (10 points) My buddy is coming over to watch the game. Unfortunately my fridge has broken down, so I have to resort to alternative measures to cool our drinks down. The drinks are initially at 20 degrees Celsius; one hour before the game starts I place the drinks in an ice box. I note that the rate of cooling of the drinks is proportional to the temperature difference between the drinks and the ice box; moreover, I observe that the proportionality constant is precisely $\frac{1}{50}$ when the units of time are minutes and the units of temperature are degrees Celsius.

   However, the icebox itself is slowing heating up. One hour before the game the icebox is at 0 degrees Celsius, but its temperature is increasing linearly at a rate of 1 degree Celsius every 10 minutes.

   Formulate and solve an initial value problem to find the temperature of the drinks when the game begins.

5. (10 total points) For the following question you may quote any formula or rule given in the Laplace transform formula sheet at the back of the exam paper.

   (a) (5 points) Compute the Laplace transform of the following function. Your answer should be a function $F(s)$.

   $$f(t) = \sin(2t) - 2\cos(t) + t^3 e^{-t}$$

   (b) (5 points) Compute the inverse Laplace transform of the following function. Your answer should be a function $f(t)$.

   $$F(s) = \frac{1 - e^{-s}}{s^2 + s}$$

6. (10 points) A certain damped oscillator obeys the following differential equation

$$y'' + 6y' + 25y = g(t),$$

where $g(t)$ is an external forcing function. For each of the following possibilities for $g(t)$, write down the form of the steady-state solution. Your answer should be in the form $Y = f(t)$, where f includes undetermined coefficients (A, B, C etc.). **You don't need to compute the actual values of these coefficients in any case**.

Each part is worth 2 points. You don't need to show your working to get full credit for this question.

(a) $g(t) = t^2 + 1$

(b) $g(t) = e^{-t} + \cos(t)$

(c) $g(t) = e^{-t}\cos(t)$

(d) $g(t) = e^{-3t}\sin(4t)$

(e) $g(t) = te^{-2t}$

7. (10 total points) A $\frac{1}{2}$ kg mass is placed on a surface and attached to a horizontal spring with spring constant β kg/s^2, where β is a positive constant. Friction acts on the mass such that when the mass is traveling at 1 m/s it experiences a frictional force of 1 Newton.

   (a) (2 points) Establish a differential equation that the mass obeys.

   (b) (5 points) For what values of β will the system will be overdamped, critically damped and underdamped respectively? Justify your answer.

   (c) (3 points) Find the value of β for which the quasi-frequency of the mass's damped oscillation is exactly 4 radians/sec. [Note: I'm referring to the *angular frequency* ω, not the cyclic frequency.]

8. (10 total points + 3 bonus points) Consider the following initial value problem:

$$y'' + 5y' + 6y = g(t), \qquad y(0) = 0, y'(0) = 0,$$

where

$$g(t) = \begin{cases} 0, & 0 \le t < 1 \\ t-1, & 1 \le t < 3 \\ 5-t, & t \ge 3 \end{cases}$$

(a) (2 points) What will the solution be for $0 \le t < 1$? Justify your answer.

(b) (3 points) Rewrite $g(t)$ in terms of Heaviside functions $u_c(t)$. Your answer should be expressible as a linear combination of $u_c(t)$'s each multiplied by some function of t. Make sure to simplify your answer.

(c) (5 points) Let $y = \phi(t)$ be the solution to this IVP. Compute the Laplace transform $\Phi(s)$ of the solution as a function of s. Be sure to simplify your answer.

[NB: you do not need to fully solve the IVP to answer this part of the question.]

[Question continued on the next page]

(d) (3 bonus points) Find the solution $y = \phi(t)$ to the IVP.

Table of Laplace Transforms

In this table, n always represents a positive integer, and a and c are real constants.

$f(t) = \mathscr{L}^{-1}[F(s)]$	$F(s) = \mathscr{L}(f(t))$	
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
t^n, n a positive integer	$\frac{n!}{s^{n+1}}$	$s > 0$
$t^n e^{ct}$, n a positive integer	$\frac{n!}{(s-c)^{n+1}}$	$s > c$
t^a, $a > -1$	$\frac{\Gamma(a+1)}{s^{a+1}}$	$s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cosh(at)$	$\frac{s}{s^2-a^2}$	$s > \lvert a \rvert$
$\sinh(at)$	$\frac{a}{s^2-a^2}$	$s > \lvert a \rvert$
$e^{ct}\cos(at)$	$\frac{s-c}{(s-c)^2+a^2}$	$s > c$
$e^{ct}\sin(at)$	$\frac{a}{(s-c)^2+a^2}$	$s > c$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	
$e^{ct}f(t)$	$F(s-c)$	
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$	$c > 0$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$	
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	