Laplace Transform Examples

Here are two examples that are nearly identical to homework 6.2/16 and 6.2/18 (only one number is changed).

1. 6.2/16 (with '5' replaced by '6'):
   Solve $y'' + 2y' + 6y = 0$ with $y(0) = 2$ and $y'(0) = -1$, using the Laplace transform.

   (a) *Laplace Transform*:
   $\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} = \mathcal{L}\{0\}$.

   (b) *Use Rules and Solve*:
   Using the derivative rules gives: $s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 2s\mathcal{L}\{y\} - 2y(0) + 6\mathcal{L}\{y\} = 0$,
   which becomes: $(s^2 + 2s + 6)\mathcal{L}\{y\} - s(2) - (-1) - 2(2) = 0$.
   Solving for $\mathcal{L}\{y\}$ gives: $\mathcal{L}\{y\} = \frac{2s+3}{s^2+2s+6}$.

   (c) *Partial Fractions*:
   Since $s^2 + 2s + 6$ does not factor, we will write it as an irreducible quadratic (*i.e.* we will complete the square): $s^2 + 2s + 6 = s^2 + 2s + 1 - 1 + 6 = (s + 1)^2 + 5$.
   Partial fractions decomposition gives
   $\frac{2s+3}{s^2+2s+6} = \frac{2s+3}{(s+1)^2+5} = \frac{A(s+1)+B}{(s+1)^2+5}$.
   And you find $2s + 3 = As + A + B$, so $A = 2$ and $B = 1$.
   Thus, $\frac{2s+3}{s^2+2s+6} = \frac{2(s+1)+1}{(s+1)^2+5} = \frac{2(s+1)}{(s+1)^2+5} + \frac{1}{(s+1)^2+5}$.

   (d) *Inverse Laplace transform*:
   The solution is: (see the inverse Laplace tranform table)
   $y(t) = 2\mathcal{L}^{-1}\{\frac{(s+1)}{(s+1)^2+5}\} + \mathcal{L}^{-1}\{\frac{1}{(s+1)^2+5}\} = 2e^{-t}\cos(\sqrt{5}t) + \frac{1}{\sqrt{5}}e^{-t}\sin(\sqrt{5}t)$.

2. 6.2/18 (with 'y(0)=1' replaced by 'y(0)=2'):
Solve $y^{(4)} - y = 0$ with $y(0) = 2$, $y'(0) = 0$, $y''(0) = 1$, $y'''(0) = 0$.

(a) *Laplace Transform*:
$\mathcal{L}\{y^{(4)}\} - \mathcal{L}\{y\} = \mathcal{L}\{0\}$.

(b) *Use Rules and Solve*:
Using the derivative rules gives: $s^4\mathcal{L}\{y\} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - \mathcal{L}\{y\} = 0$,
which becomes: $(s^4 - 1)\mathcal{L}\{y\} - 2s^3 - s = 0$.
Solving for $\mathcal{L}\{y\}$ gives: $\mathcal{L}\{y\} = \frac{2s^3+s}{s^4-1}$.

(c) *Partial Fractions*:
Factoring gives: $s^4 - 1 = (s^2 - 1)(s^2 + 1) = (s - 1)(s + 1)(s^2 + 1)$.
Partial fractions decomposition gives
$\frac{2s^3+s}{(s-1)(s+1)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}$.
The cover up method gives $A = \frac{2(1)^3+1}{(1+1)(1^1+1)} = \frac{3}{4}$ and $B = \frac{2(-1)^3-1}{((-1)-1)((-1)^2+1)} = \frac{3}{4}$.
Then expanding gives
$2s^3 + s = \frac{3}{4}(s + 1)(s^2 + 1) + \frac{3}{4}(s - 1)(s^2 + 1) + (Cs + D)(s^2 - 1)$, so
$2s^3 + s = \frac{3}{4}(s^3 + s^2 + s + 1) + \frac{3}{4}(s^3 - s^2 + s - 1) + Cs^3 + Ds^2 - Cs - D$, which we can regroup to get
$2s^3 + s = (\frac{3}{4} + \frac{3}{4} + C)s^3 + Ds^2 + (\frac{3}{4} + \frac{3}{4} - C)s - D$.
Thus, $2 = \frac{3}{4} + \frac{3}{4} + C$, so $C = 1/2$,
Also $D = 0$.
Thus, $\frac{2s^3+1}{s^4-1} = \frac{3/4}{s-1} + \frac{3/4}{s+1} + \frac{1/2s}{s^2+1}$.

(d) *Inverse Laplace transform*:
The solution is: $y(t) = \frac{3}{4}\mathcal{L}^{-1}\{\frac{1}{s-1}\} + \frac{3}{4}\mathcal{L}^{-1}\{\frac{1}{s+1}\} + \frac{1}{2}\mathcal{L}^{-1}\{\frac{s}{s^2+1}\}$.
Thus, $y(t) = \frac{3}{4}e^t + \frac{3}{4}e^{-t} + \frac{1}{2}\cos(t)$.