3.1: Homogeneous Constant Coefficient 2nd Order

Some Observations and Motivations:

1. For equations of the form $ay'' + by' + cy = 0$, we are looking for a function that 'cancels' with itself if you take its first and second derivatives and add up $ay'' + by' + cy$. This means that the derivatives of y will have to look similar to y in some way. (You should be thinking of functions like $y = ke^{rt}$, $y = k\cos(rt)$ and $y = k\sin(rt)$).

2. In section 3.1, we are going to try to see if we can find solutions of the form $y = e^{rt}$ for some constant r. If $y = e^{rt}$ is a solution, then that means it works in the differential equation. Taking derivatives (using the chain rule), you get $y = e^{rt}$, $y' = re^{rt}$, and $y'' = r^2e^{rt}$. And if you substitute these into the differential equation you get

$$ay'' + by' + cy = 0 \quad \text{which becomes} \quad ar^2e^{rt} + bre^{rt} + ce^{rt} = e^{rt}(ar^2 + br + c) = 0.$$

3. We are looking for a function $y = e^{rt}$ that makes this true for all values of t. Since e^{rt} is never zero, we are looking for values of r that make $ar^2 + br + c = 0$.

4. You already do have some experience with second order equations. Consider $\frac{d^2y}{dt^2} = -9.8$. This is second order but it doesn't involve y' or y, so you can integrate twice to get $y = -4.9t^2 + c_1t + c_2$. Notice that you get **two constants** of integration. We will see in section 3.2 that is true in general for second order equations, we will get two constants in our general solutions.

Definitions and Two Real Roots Method:

1. For the equation $ay'' + by' + cy = 0$, we define the **characteristic equation** to be $ar^2 + br + c = 0$.

2. The **roots** of the characteristic equation are the solutions $r_1 = \frac{-b-\sqrt{b^2-4ac}}{2a}$ and $r_2 = \frac{-b+\sqrt{b^2-4ac}}{2a}$. There are three cases:

   - if $b^2 - 4ac > 0$, then you get two real roots. (Section 3.1 is about this case)
   - if $b^2 - 4ac = 0$, then you get one (repeated) root. (Section 3.4)
   - if $b^2 - 4ac < 0$, then you get no real roots, but two complex (imaginary) roots. (Section 3.3)

3. If there are two real roots, r_1 and r_2, then that means $y_1(x) = e^{r_1x}$ and $y_2(x) = e^{r_2x}$ are both solutions. All other solutions can be written in the form

$$y = c_1e^{r_1x} + c_2e^{r_2x},$$

   for some constants c_1 and c_2. We call this the general solution.
   We will discuss the 'why' all solutions are in this form in section 3.2.

Examples:

1. Give the general solution to $y'' - 7y' + 10y = 0$.

   Solution: The equation $r^2 - 7r + 10 = (r-5)(r-2) = 0$ has roots $r_1 = 2$ and $r_2 = 5$.
   The general solution is $y = c_1e^{2t} + c_2e^{5t}$.

2. Give the general solution to $y'' + 4y' = 0$.

   Solution: The equation $r^2 + 4r = r(r+4) = 0$ has roots $r_1 = -4$ and $r_2 = 0$.
   The general solution is $y = c_1e^{-4t} + c_2$.

Examples with initial conditions:

1. Solve $y'' - 9y = 0$ with $y(0) = 2$ and $y'(0) = -12$.

   Solution: The equation $r^2 - 9 = (r+3)(r-3) = 0$ has roots $r_1 = -3$ and $r_2 = 3$.
   The general solution is $y = c_1e^{-3t} + c_2e^{3t}$. Note that $y' = -3c_1e^{-3t} + 3c_2e^{3t}$.
   Substituting in the initial condition gives

   $$\begin{array}{lll} y(0) = 2 & \Rightarrow & c_1 + c_2 = 2 \\ y'(0) = -12 & \Rightarrow & -3c_1 + 3c_2 = -12 \Rightarrow -c_1 + c_2 = -4 \end{array}$$

   Note that we divided equation (ii) by 3. Now we combine and simplify. Adding the equations gives $2c_2 = -2$, so $c_2 = -1$. And using either equation gives $c_1 = 3$.
   Thus, the solution is $y(t) = 3e^{-3t} - e^{3t}$.

2. Solve $y'' - 4y' - 5y = 0$ with $y(0) = 7$ and $y'(0) = 1$.

   Solution: The equation $r^2 - 4r - 5 = (r+1)(r-5) = 0$ has roots $r_1 = -1$ and $r_2 = 5$.
   The general solution is $y = c_1e^{-t} + c_2e^{5t}$. Note that $y' = -c_1e^{-t} + 5c_2e^{5t}$.
   Substituting in the initial condition gives

   $$\begin{array}{lll} y(0) = 7 & \Rightarrow & c_1 + c_2 = 7 \\ y'(0) = 1 & \Rightarrow & -c_1 + 5c_2 = 1 \end{array}$$

   Now we combine and simplify. Adding the equations gives $6c_2 = 8$, so $c_2 = \frac{4}{3}$. And using either equation gives $c_1 = 7 - \frac{4}{3} = \frac{17}{3}$. Thus, the solution is $y(t) = \dfrac{17}{3}e^{-t} + \dfrac{4}{3}e^{5t}$.