3.4: Homogeneous Constant Coefficient 2nd Order (Repeated Roots)

Just like in my 3.3 review, let me give away the 'punchline'. In other words, let me show how easy it is to solve these problems once you know the general result, then we'll discuss the theoretical underpinnings:

Solutions for the One Real Root Case:
If $ar^2 + br + c = 0$ has only one real roots r, then the general solution to $ay'' + by' + cy = 0$ is given by

$$y(t) = c_1 e^{rt} + c_2 t e^{rt}.$$

Examples:

1. Give the general solution to $y'' + 10y' + 25y = 0$.

   Solution: The equation $r^2 + 10r + 25 = (r+5)^2 = 0$ has only one root $r = -5$.
   The general solution is $y = c_1 e^{-5t} + c_2 t e^{-5t}$.

2. Give the general solution to $y'' - 6y' + 9y = 0$.

   Solution: The equation $r^2 - 6r + 9 = (r-3)^2 = 0$ has only one root $r = 3$.
   The general solution is $y = c_1 e^{3t} + c_2 t e^{3t}$.

Examples with initial conditions:

1. Solve $y'' + 4y' + 4 = 0$ with $y(0) = 2$ and $y'(0) = 5$.

   Solution: The equation $r^2 + 4r + 4 = (r+2)^2 = 0$ has only one root $r = -2$.
   The general solution is $y = c_1 e^{-2t} + c_2 t e^{-2t}$.
   Note that $y' = -2c_1 e^{-2t} + c_2(e^{-2t} - 2te^{-2t})$.

   $$y(0) = 2 \quad \Rightarrow \quad c_1 + 0 = 2 \Rightarrow c_1 = 2$$
   $$y'(0) = 5 \quad \Rightarrow \quad -2c_1 + c_2 = 5 \Rightarrow c_2 = 9$$

   Thus, the solution is $y(t) = 2e^{-2t} + 9te^{-2t}$.

2. Solve $y'' - 8y' + 16y = 0$ with $y(0) = -3$ and $y'(0) = 1$.

   Solution: The equation $r^2 - 8r + 16 = (r-4)^2 = 0$ has only one root $r = 4$.
   The general solution is $y = c_1 e^{4t} + c_2 t e^{4t}$.
   Note that $y' = 4c_1 e^{4t} + c_2(e^{4t} + 4te^{4t})$.

   $$y(0) = -3 \quad \Rightarrow \quad c_1 + 0 = -3 \Rightarrow c_1 = -3$$
   $$y'(0) = 4 \quad \Rightarrow \quad 4c_1 + c_2 = 1 \Rightarrow c_2 = 13$$

   Thus, the solution is $y(t) = -3e^{4t} + 13te^{4t}$.

Observations and Motivation:
As we discussed in class, we started with one solution $y_1(t) = e^{rt}$ and we needed to find another. We made the educated guess that a second solution might have the form $y(t) = u(t)e^{rt}$. By differentiating and substituting, we found that this indeed gave a solution when $u(t) = t$. This is called the method of reduction of order. This is a **general method** that works for linear questions (even for higher order). It takes one known solution and attempts to find other solutions. Here is a more general discussion:

Method of Reduction of Order:
If $y = y_1(t)$ is one known solution to $y'' + p(t)y' + q(t)y = 0$, then the method of reduction attempts to find another solution as follow:

1. Write $y = u(t)y_1(t)$. You will attempt to find $u(t)$.

2. Find $y' = u'(t)y_1(t) + u(t)y_1'(t)$ and $y'' = u''(t)y_1(t) + 2u'(t)y_1'(t) + u(t)y_1''(t)$.

3. Substitute into $y'' + p(t)y' + q(t)y = 0$ and simplify.

4. You now will have an equation in the form $y_1(t)u'' + (2y_1'(t) + p(t)y_1(t))u' = 0$. Note that if you write $v(t) = u'(t)$, then this equation is the first order equation $y_1(t)\frac{dv}{dt} + (2y_1'(t) + p(t)y_1(t))v = 0$. Solve this first order equation! From this get $u'(t)$.

5. Integrate to get $u(t)$. This will involve constants of integration. For any choice of those constants, the following with be a solution: $y_2(t) = u(t)y_1(t)$. (We look for a solution that is indeed different from the first).

Side Note: With a bit of general work with integrating factors and some simplification, you can find that the solution of $y_1(t)\frac{dv}{dt} + (2y_1'(t) + p(t)y_1(t))v = 0$ will look like $v(t) = \frac{1}{y_1(t)}e^{-\int p(t)\,dt}$ is a solution to this first order equation. Since $u'(t) = v(t)$, that means that $u(t) = \int v(t)\,dt = \int \frac{1}{y_1(t)}e^{-\int p(t)\,dt}\,dt$. This is a compact integral formula for the final form of $u(t)$. But for the problems we do, it will be just as easy to follow the procedure above.

Examples of Reduction of Order are on the next page.

Examples:

1. First, let's redo the example of a repeated root:
   Assume you want to solve $y'' + 10y' + 25y = 0$ and you know one solution is $y_1(t) = e^{-5t}$.

   Solution:

   (a) Let $y = u(t)e^{-5t} = ue^{-5t}$,

   (b) Then $y' = u'e^{-5t} - 5ue^{-5t} = (u' - 5u)e^{-5t}$ and
   $$y'' = (u'' - 5u')e^{-5t} - 5(u' - 5u)e^{-5t} = (u'' - 10u' + 25u)e^{-5t}$$

   (c) Substiting gives $y'' + 10y' + 25y = (u'' - 10u' + 25u)e^{-5t} + 10(u' - 5u)e^{-5t} + 25ue^{-5t} = 0$,
   which simplifies to $u'' - 10u' + 25u + 10u' - 15u + 25u = u'' = 0$
   (Note: The u's will always cancel here! In this case, the u' also cancelled, but that won't always happen)

   (d) Letting $v = u'$, we see that we are looking at a first order equation $v' = 0$, which has solution
   $u'(t) = v(t) = a_1$ (a constant).
   Integrating again we get $u(t) = a_1 t + a_2$.

   (e) Thus, any answer in the form $y = u(t)e^{-5t} = a_1 te^{-5t} + a_2 e^{-5t}$ is also a solution. We can see
   a 'new' solution here is $y_2(t) = te^{-5t}$.

   The general answer is $y = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{-5t} + c_2 te^{-5t}$.

2. Another example:
   Assume you need to solve the differential equation $t^2 y'' - 6ty' + 12y = 0$. After some experimentation, you find one solution is $y_1(t) = t^3$. Find the general solution.

   Solution:

   (a) Let $y = ut^3$,

   (b) Then $y' = u't^3 + 3ut^2$ and
   $$y'' = u''t^3 + 3u't^2 + 3u't^2 + 6ut = u''t^3 + 6u't^2 + 6ut$$

   (c) Substiting gives $t^2(u''t^3 + 6u't^2 + 6ut) - 6t(u't^3 + 3ut^2) + 12ut^3 = 0$,
   which expands to $t^3 u'' + 2t^2 u' - u't^3 - ut^2 - 4u't^2 - 4ut + t^2 u + 4tu = 0$.
   which simplies to $t^5 u'' = 0$, so we again need $u'' = 0$.

   (d) Again we get $u(t) = a_1 t + a_2$.

   (e) Thus, any answer in the form $y = u(t)t^3 = (a_1 t + a_2)t^3 = a_1 t^4 + a_2 t^3$ is also a solution. We
   can see a 'new' solution here is $y_2(t) = t^4$.

   The general answer is $y = c_1 y_1(t) + c_2 y_2(t) = c_1 t^3 + c_2 t^4$.

3. Another 'messier' example:

Assume you need to solve the differential equation $t^2 y'' - t(t+4)y' + (t+4)y = 0$. After some experimentation, you find one solution is $y_1(t) = t$. Find the general solution.

Solution:

(a) Let $y = ut$,

(b) Then $y' = u't + u$ and $y'' = u''t + u' + u' = u''t + 2u'$

(c) Substiting gives $t^2(u''t + 2u') - t(t+4)(u't + u) + (t+4)ut = 0$,
which expands to $t^3 u'' + 2t^2 u' - u't^3 - ut^2 - 4u't^2 - 4ut + t^2 u + 4tu = 0$.
which simplies to $t^3 u'' - (t^3 + 2t^2)u' = 0$.
Dividing by t^3 we see we need to solve $u'' - \left(1 + \frac{2}{t}\right)u' = 0$.

(d) Letting $v = u'$, we see that we are looking at a first order equation $v' - \left(1 + \frac{2}{t}\right)v = 0$, which has solution $u'(t) = v(t) = a_1 t^2 e^t$ (do this by integrating factors or separation!)
Integrating again (by parts twice) we get $u(t) = a_1 e^t(t^2 - 2t + 2) + a_2$.

(e) Thus, any answer in the form:
$y = u(t)t = (a_1 e^t(t^2 - 2t + 2) + a_2)t = a_1 te^t(t^2 - 2t + 2) + a_2 t$ is also a solution.
We can see a 'new' solution here is $y_2(t) = te^t(t^2 - 2t + 2) = e^t(t^3 - 2t^2 + 2t)$.

The general answer is $y = c_1 y_1(t) + c_2 y_2(t) = c_1 t + c_2 e^t(t^3 - 2t^2 + 2t)$.

4. A third order example!

Assume you need to solve the **third order** differential equation $y''' - 7y' + 6y = 0$. After some experimentation, you find one solution is $y_1(t) = e^t$. Find the general solution (you need three different solution).

Solution:

(a) Let $y = ue^t$,

(b) Then $y' = u'e^t + ue^t = (u' + u)e^t$, $y'' = (u'' + u')e^t + (u' + u)e^t = (u'' + 2u' + u)e^t$, and
$y''' = (u''' + 2u'' + u')e^t + (u'' + 2u' + u)e^t = (u''' + 3u'' + 3u' + u)e^t$

(c) Substiting gives $(u''' + 3u'' + 3u' + u)e^t - 7(u' + u)e^t + 6ue^t = 0$,
which expands to $u''' + 3u'' + 3u' + u - 7u' - 7u + 6u = 0$.
which simplies to $u''' + 3u'' - 4u' = 0$. (Note: that u is gone)

(d) Letting $v = u'$, we see that we are looking at a **second order** equation $v'' + 3v' - 4v = 0$ (we have reduce the order).
Using our current methods, we can solve this by getting the characteristic equation $r^2 + 3r - 4 = (r+4)(r-1) = 0$ so the solution is $u'(t) = v(t) = a_1 e^{-4t} + a_2 e^t$.
Integrating again gives $u(t) = \frac{a_1}{-4}e^{-4t} + a_2 e^t + a_3$ (let's redefine $a_1 = a_1/(-4)$ from here on out since it is just constant)

(e) Thus, any answer in the form:
$y = u(t)e^t = (a_1 e^{-4t} + a_2 e^t + a_3)e^t = a_1 e^{-3t} + a_2 e^{2t} + a_3 e^t$ is also a solution.
We can see two 'new' solutions here are $y_2(t) = e^{-3t}$ and $y_3(t) = e^{2t}$.

The general answer is $y = c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t) = c_1 e^t + c_2 e^{-3t} + c_3 e^{2t}$.