3.7: Analyzing Mechanical and Electrical Vibrations (Free Vibrations)

This review just discusses analysis of these applications. For the set up, read the 3.7 and 3.8 applications review. In 3.7 we are considering, 'free vibrations' which means there is no forcing. In other words, we are considering the homogeneous equation with $F(t) = 0$.

For an object attached to a spring that is not being forced, we found that the displacement from rest, $u(t)$, at time t satisfies:

$$mu'' + \gamma u' + ku = 0,$$

where m is the mass, γ is the damping (friction) constant, and k is the spring constant (all these constants are positive).

We will analyze different cases:

Undamped Free Vibrations: (The $\gamma = 0$ case)
If we assume there is no friction (or that the friction is small enough to be negligible), then we are taking $\gamma = 0$. In which case we get:

$$mu'' + ku = 0.$$

The roots of $mr^2 + k = 0$ are $r = \pm i\sqrt{k/m}$, so the general solution is

$$u(t) = c_1 \cos(\sqrt{k/m}\, t) + c_2 \sin(\sqrt{k/m}\, t).$$

Using the facts from my review sheet on waves (namely, $R = \sqrt{c_1^2 + c_2^2}$, $c_1 = R\cos(\delta)$, and $c_2 = R\sin(\delta)$), we can rewrite this in the form

$$u(t) = R\cos(\omega_0 t - \delta),$$

where $\omega_0 = \sqrt{k/m}$.
Thus, the solution is a cosine wave with the following properties:

- The **natural frequency** is $\omega_0 = \sqrt{k/m}$ radians/second.
- The **period** (or **wavelength**) is $T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{m/k}$ seconds/wave
(this is the time from peak-to-peak or valley-to-valley).
- The **amplitude** is $R = \sqrt{c_1^2 + c_2^2}$, which will depend on initial conditions.
- The **phase angle** is δ which is the starting angle, which also depends on initial conditions.

Damped Free Vibrations: (The $\gamma > 0$ case)
If $\gamma > 0$, then we have

$$mu'' + \gamma u' + ku = 0.$$

The roots of $mr^2 + \gamma r + k = 0$ are $r = -\frac{\gamma}{2m} \pm \frac{1}{2m}\sqrt{\gamma^2 - 4mk}$. Three different things can happen here:

1. If $\gamma^2 - 4km > 0$, then there are two real roots that are both negative.
   The solution looks like $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.
   The condition simplifies to $\gamma > 2\sqrt{km}$.
   In this case we say the systems is **overdamped**.

2. If $\gamma^2 - 4km = 0$, then there is one repeated root that is negative.
   The solution looks like $y = c_1 e^{rt} + c_2 t e^{rt}$.
   The condition simplifies to $\gamma = 2\sqrt{km}$.
   In this case we say the systems is **critically damped**.

3. If $\gamma^2 - 4km < 0$, then there are two complex roots with $\lambda = -\frac{\gamma}{2m}$ and $\omega = \mu = \frac{\sqrt{4mk-\gamma^2}}{2m}$.
   The solution looks like $y = e^{\lambda t}(c_1 \cos(\mu t) + c_2 \sin(\mu t))$.
   The condition simplifies to $\gamma < 2\sqrt{km}$. In this case, we get oscillations where the amplitude goes to zero. We can analyze the wave part of this last case like we did before.
   The expression $c_1 \cos(\mu t) + c_2 \sin(\mu t)$ can be rewritten as $R\cos(\mu t - \delta)$,
   where $R = \sqrt{c^1 + c_2^2}$, $c_1 = R\cos(\delta)$ and $c_2 = R\sin(\delta)$.
   Thus, in this case, the general answer can be written as

   $$u(t) = Re^{\lambda t}\cos(\mu t - \delta),$$

   where

   - The **quasi frequency** is $\mu = \frac{\sqrt{4mk-\gamma^2}}{2m}$ radians/second.
   - The **quasi period** is $T = \frac{2\pi}{\mu} = 2\pi\frac{2m}{\sqrt{4mk-\gamma^2}}$ seconds/wave.
   - The **amplitude** is $Re^{\lambda t}$, which will always go to zero as $t \to \infty$.

   Note: If the damping is small, then γ is close to zero. Notice that the formulas above for quasi frequency and quasi period become the same as the frequency and period when $\gamma = 0$. So we get similar frequencies and periods between small damping and no damping.