1. (8 points) The differential equation of a given mass-spring system is $4u'' + \gamma u' + 9u = 0$ with initial conditions $u(0) = 2$ and $u'(0) = 0$. You are told that the system is **critically damped**. Solve the initial value problem to find $u(t)$. (Hint: First find γ).

CRITICALLY DAMPED $\Rightarrow$ $\gamma = 2\sqrt{mk} = 2\sqrt{4 \cdot 9} = 12$

CHARACTERISTIC EQUATION: $4r^2 + 12r + 9 = 0$

$(2r+3)^2 = 0$

$r = -3/2$

GENERAL SOL'N: $u(t) = c_1 e^{-\frac{3}{2}t} + c_2 t e^{-\frac{3}{2}t}$

INITIAL CONDITIONS:

$u(0) = c_1 = 2$

$u'(t) = -\frac{3}{2} c_1 e^{-3/2 t} + c_2 e^{-\frac{3}{2}t} - \frac{3}{2} c_2 e^{-\frac{3}{2}t}$

$u'(0) = 0 \Rightarrow -\frac{3}{2}\underset{\uparrow \atop 2}{c_1} + c_2 = 0 \Rightarrow c_2 = 3$

$$\boxed{u(t) = 2e^{-\frac{3}{2}t} + 3te^{-\frac{3}{2}t}}$$

2. (10 pts) **Parts (a) and (b) below are independent!**

(a) A 5 pound object stretches a spring 3 inches beyond its natural length (and is at rest). The damping force is 2 lbs when the upward velocity is 10 ft/s. There is no external forcing. Initially, the mass is pulled downward 6 in and given an initial upward velocity of 1 ft/s. Set up the differential equation AND initial conditions for the displacement $u(t)$. (DO NOT SOLVE)

$$\left.\begin{aligned} w &= 5 \text{ lbs} \\ L &= \tfrac{3}{12} = \tfrac{1}{4} \text{ ft} \end{aligned}\right\} \quad w - kL = 0 \Rightarrow k = \frac{w}{L} = \frac{5}{1/4} = 20 \ \frac{\text{lbs}}{\text{ft}}$$

$$mg = w \Rightarrow m = \frac{w}{g} = \frac{5}{32} \ \frac{\text{lbs}}{\text{ft/s}^2}$$

$$F_d = -\gamma u' \Rightarrow \quad \gamma = \frac{-F_d}{u'} = \frac{-2}{-10} = \frac{1}{5} \ \frac{\text{lbs}}{\text{ft/s}}$$

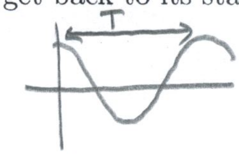
$$6 \text{ in} = \tfrac{1}{2} \text{ ft}$$

$$\boxed{\begin{gathered} \frac{5}{32}u'' + \frac{1}{5}u' + 20u = 0 \\ u(0) = \frac{1}{2}, \quad u'(0) = -1 \end{gathered}}$$

(b) For a different mass-spring system the differential equation is $2u'' + 14u = F(t)$, where $F(t)$ is the forcing function and t is in seconds.

i. Assume there is no forcing ($F(t) = 0$). The mass is pulled down to some starting point. The mass is released. How long will it take to first get back to its starting point?

$$2r^2 + 14 = 0 \Rightarrow \quad r = \pm\sqrt{7}\, i$$



$$\omega_0 = \sqrt{7}$$

$$\boxed{T = \frac{2\pi}{\sqrt{7}} \text{ seconds}}$$

ii. Let ω_0 be the natural frequency of the unforced system. Assume $F(t) = 5\cos(\omega t)$ where ω is the frequency of the forcing. There is a significant difference in long-term behavior of the solutions depending on whether $\omega = \omega_0$ or $\omega \neq \omega_0$. Briefly in words, give the name of this behavior AND specifically describe the key difference in behavior. (You don't have to solve anything here, one to two sentences is all you need).

$\omega = \omega_0 \Rightarrow$ $\boxed{\text{RESONANCE}}$ (since $\gamma = 0$)

The amplitude will increase without bound.

$\omega \neq \omega_0 \Rightarrow$ The amplitude is bounded.

3. (10 pts) The charge $Q(t)$ at time t on the capacitor in a particular RLC circuit satisfies

$$Q'' + Q' + \frac{5}{4}Q = \frac{37}{4}\cos(2t).$$

A particular solution is given by $Y(t) = -\frac{11}{5}\cos(2t) + \frac{8}{5}\sin(2t)$. (You do not have to find this!)

(a) Give the general solution for $Q(t)$.

$$r^2 + r + \tfrac{5}{4} = 0 \Rightarrow r = \frac{-1}{2} \pm \frac{1}{2}\sqrt{1 - 4\cdot\tfrac{5}{4}}$$

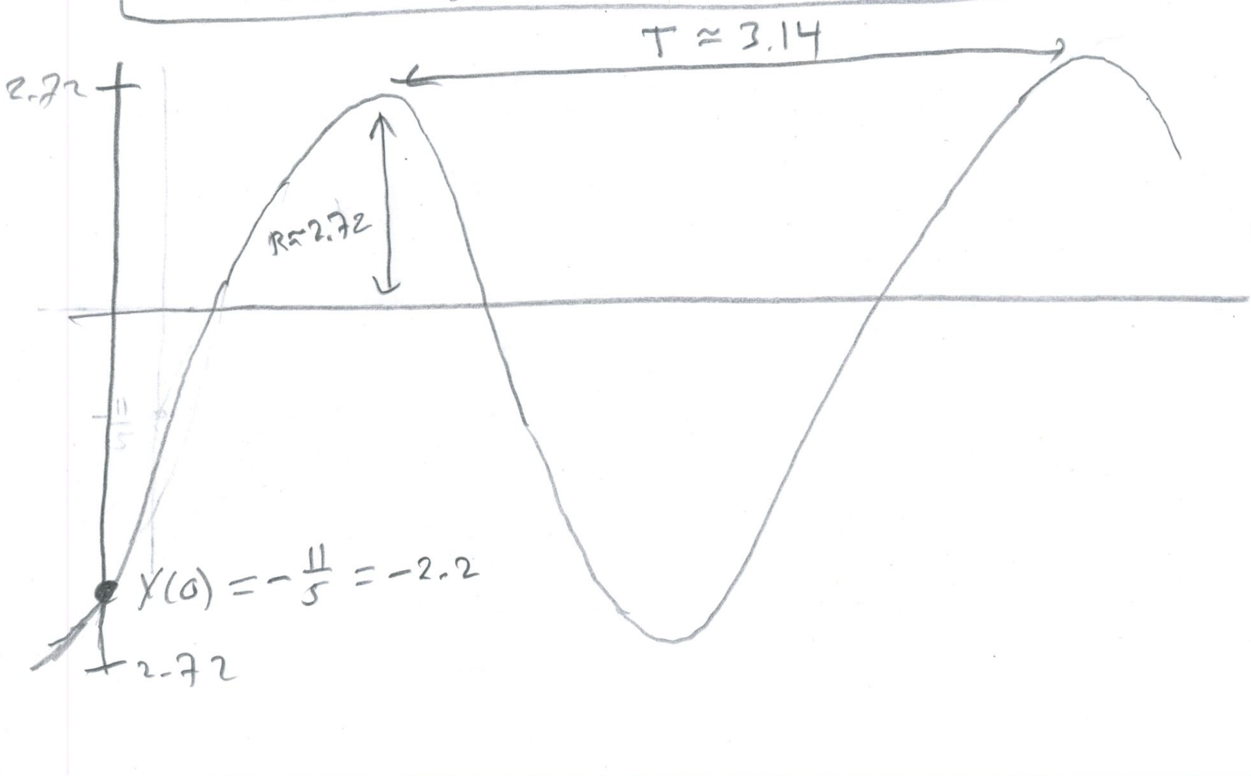
$$r = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{-4} = -\tfrac{1}{2} \pm i$$

$$\lambda = -\tfrac{1}{2},\ \mu = 1$$

$$\boxed{Q(t) = c_1 e^{-\frac{1}{2}t}\cos(t) + c_2 e^{-\frac{1}{2}t}\sin(t) - \frac{11}{5}\cos(2t) + \frac{8}{5}\sin(2t)}$$

(b) Draw a rough sketch of the steady state response.
Find and clearly label the amplitude and period. Also label the y-intercept.

- $Y(0) = -\frac{11}{5} = -1.2$, ALSO NOTE $Y'(0) = \frac{16}{5} = 3.2$ (Starts going upward)
- $\omega = 2 \Rightarrow \boxed{T = \frac{2\pi}{2} = \pi \approx 3.14}$ ← PERIOD
- $\boxed{R = \sqrt{\left(-\frac{11}{5}\right)^2 + \left(\frac{8}{5}\right)^2} = \frac{1}{5}\sqrt{185} \approx 2.72}$ ← AMPLITUDE



4. (12 pts) Note: The equation $r^2 - r - 2 = 0$ has the roots $r_1 = 2$ and $r_2 = -1$.

(a) Use undetermined coefficients to find a particular solution to $y'' - y' - 2y = 5 + 3e^{2t}$.

$Y(t) = A + Bte^{2t}$

NEED THIS BECAUSE e^{2t} IS A HOMOGENEOUS SOL'N

$Y'(t) = Be^{2t} + 2Bte^{2t}$

$Y''(t) = 2Be^{2t} + 2Be^{2t} + 4Bte^{2t} = 4Be^{2t} + 4Bte^{2t}$

$$\overbrace{4Be^{2t} + 4Bte^{2t}}^{y''} - \overbrace{Be^{2t} - 2Bte^{2t}}^{y'} - \overbrace{2A - 2Bte^{2t}}^{2y} \stackrel{?}{=} 5 + 3e^{2t}$$

$$-2A + 3Be^{2t} \stackrel{?}{=} 5 + 3e^{2t}$$

$$\Rightarrow \begin{cases} -2A = 5 \Rightarrow A = -5/2 \\ 3B = 3 \Rightarrow B = 1 \end{cases}$$

$$\boxed{Y(t) = -\frac{5}{2} + te^{2t}}$$

(b) Use undetermined coefficients to find a particular solution to $y'' - y' - 2y = \cos(t)$.

$Y(t) = A\cos(t) + B\sin(t)$

$Y'(t) = -A\sin(t) + B\cos(t)$

$Y''(t) = -A\cos(t) - B\sin(t)$

$$\overbrace{-A\cos(t) - B\sin(t)}^{y''} - \overbrace{+A\sin(t) - B\cos(t)}^{y'} - \overbrace{2A\cos(t) - 2B\sin(t)}^{2y} \stackrel{?}{=} \cos(t)$$

$$(-3A - B)\cos(t) + (A - 3B)\sin(t) \stackrel{?}{=} \cos(t)$$

$$\Rightarrow \begin{cases} -3A - B = 1 \\ A - 3B = 0 \Rightarrow A = 3B \end{cases}$$

$-9B - B = 1 \Rightarrow B = -\frac{1}{10}$

$A = -\frac{3}{10}$

$$\boxed{Y(t) = -\frac{3}{10}\cos(t) - \frac{1}{10}\sin(t)}$$

5. (10 pts) Consider the differential equation

$$y'' - \frac{6}{t}y' + \frac{6}{t^2}y = 5t^3 \quad \text{with} \quad t > 0.$$

The function $y_1(t) = t$ is a solutions to the corresponding homogeneous equation. Use reduction of order to find the general solution.

$y = ut$

$y' = u't + u$

$y'' = u''t + u' + u' = u''t + 2u'$

$$y'' - \frac{6}{t}y' + \frac{6}{t^2}y = 5t^3$$

$$\underbrace{u''t + 2u'}_{} - \frac{6}{t}\underbrace{(u't + u)}_{} + \frac{6}{t^2}ut \overset{?}{=} 5t^3$$

$$u''t + 2u' - 6u' \cancel{- \frac{6}{t}u + \frac{6}{t}u} \overset{?}{=} 5t^3$$

$$t u'' - 4u' = 5t^3$$

FIRST ORDER! $\Rightarrow$ $u'' - \frac{4}{t}u' = 5t^2$

$\mu(t) = e^{\int -4/t\,dt} = e^{-4\ln(t)} = t^{-4}$

$$\underbrace{t^{-4}u'' - 4t^{-5}u'}_{} = 5t^{-2}$$

$$\frac{d}{dt}\left(t^{-4}u'\right) = 5t^{-2}$$

$$t^{-4}u' = -5t^{-1} + a_1$$

$$u'(t) = -5t^3 + a_1 t^4$$

$$u(t) = -\frac{5}{4}t^4 + \frac{a_1}{5}t^5 + a_2$$

$y(t) = u(t)t = -\frac{5}{4}t^5 + \frac{a_1}{5}t^6 + a_2 t$

LET $c_1 = a_2$

$c_2 = \frac{a_1}{5}$

$$\boxed{y(t) = c_1 t + c_2 t^6 - \frac{5}{4}t^5}$$