Math 307 Section F
Spring 2013
Final Exam
May 22, 2013
Time Limit: 2 Hours

Name (Print): ____________________

Student ID: ____________________

This exam contains 15 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam. However, you may use a single, handwritten, one-sided notesheet and a *basic* calculator.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

1. Solve the following initial value problems

   (a) (5 points) Solve the following initial value problems:

$$\frac{dy}{dx} = e^x y - x^2 y, \quad y(0) = 1$$

   (b) (5 points)

$$y' = 3y + e^t, \quad y(0) = 1/2$$

Solution.

(a) This equation is separable! In particular, by dividing both sides by y, we get

$$\frac{1}{y}\frac{dy}{dx} = e^x - x^2.$$

Then integrating both sides of this equation:

$$\int \frac{1}{y} dy = \int e^x - x^2 dx,$$

which gives us

$$\ln(y) = e^x - \frac{1}{3}x^3 + C.$$

Finally, by setting $x = 0$ and $y = 1$:

$$\ln(1) = e^0 - \frac{1}{3}x^3 + C,$$

which tells us $C = -1$. Therefore the solution of the initial value problem is given by

$$\ln(y) = e^x - \frac{1}{3}x^3 - 1.$$

Of course, you could finish this off by completely solving for y, but it isn't necessary.

(b) This equation is a first order linear equation, pointing us towards using either an integrating factor or the method of variation of parameters. Let's use the first one. An integrating factor for this equation is easily found to be $\mu(t) = e^{-3t}$. Multiplying the original equation by this integrating factor, we obtain

$$e^{-3t}y' = 3e^{-3t}y + e^{-2t}.$$

Moving the y and y' terms to the left hand side, we get

$$e^{-3t}y' - 3e^{-3t}y = e^{-2t}.$$

At this point (as per usual) we see that we can group the terms on the left hand side to find

$$(e^{-3t}y)' = e^{-2t}$$

Then integrating both sides with respect to t, we get:

$$e^{-3t}y = -\frac{1}{2}e^{-2t} + C$$

and pluggin in $x = 0$ and $y = 1/2$, we find

$$\frac{1}{2} = -\frac{1}{2} + C,$$

so that $C = 1$ and therefore

$$y = -\frac{1}{2}e^t + e^{3t}.$$

2. **Propose a Solution Section!**

Directions: The "Propose a Solution" section consists of five linear nonhomogeneous equations. For each of these equations, write down the type of function y (with undetermined coefficients) you would try, in order to get a particular solution. *You do NOT need to solve the equations* For example, if the equation were

$$y'' + 2y' + y = e^t,$$

a *correct answer* would be

$$y = Ae^t,$$

and *incorrect answers* would include

$$y = (At + B)e^t, \quad y = At^2 e^{2t}, \quad y = Ae^{3t}, \quad y = A\pi^t$$

Each part is worth 2pts:

(a) (2 points)

$$y'' + 3y' + 2y = t^5 e^{4t}$$

(b) (2 points)

$$y'' + 3y' + 2y = (t + 1)e^{-2t}$$

(c) (2 points)

$$y'' - 2y' + y = 2te^t$$

(d) (2 points)

$$y'' + 2y' + y = 4t^2 e^t$$

(e) (2 points)

$$y'' - 2y' = te^{2t} - 7e^{2t}$$

Solution.

(a) $y_p = (At^5 + Bt^4 + Ct^3 + Dt^2 + Et + F)e^{4t}$

(b) $y_p = (At^2 + Bt)e^{-2t}$

(c) $y_p = (At^3 + Bt^2)e^t$

(d) $y_p = (At^2 + Bt + C)e^t$

(e) $y_p = (At^2 + Bt)e^{2t}$

3. (10 points) Solve the equation

$$(3xy + y^2)dx + (x^2 + xy)dy = 0$$

by finding an integrating factor.

Solution. We try an integrating factor of the form $\mu = \mu(x)$. For this to be an integrating factor, the equation

$$\overbrace{(3xy + y^2)\mu(x)}^{M(x,y)} + \overbrace{(x^2 + xy)\mu(x)}^{N(x,y)}\, y' = 0$$

must be exact, meaning that $M_y = N_x$. We calculate

$$M_y(x, y) = (3x + 2y)\mu(x)$$

and

$$N_x(x, y) = (2x + y)\mu(x) + (x^2 + xy)\mu'(x)$$

meaning that

$$(3x + 2y)\mu(x) = (2x + y)\mu(x) + (x^2 + xy)\mu'(x).$$

After some algebraic simplification, this becomes

$$(x + y)\mu(x) = (x^2 + xy)\mu'(x).$$

Then the important thing to notice is that $(x^2 + xy) = x(x + y)$, so that by dividing both sides by $x + y$ we find

$$\mu(x) = x\mu'(x).$$

A solution to this is $\mu(x) = x$, which gives us our integrating factor. It follows that the equation

$$\overbrace{(3x^2y + xy^2)}^{M(x,y)} + (x^3 + x^2y)^{N(x,y)} y' = 0,$$

is exact. Therefore there must exist a function $\psi(x, y)$ satisfying $\psi_x = M$ and $\psi_y = N$. This means that for example

$$\psi_y = (x^3 + x^2y),$$

and therefore

$$\psi = \int (x^3 + x^2y)\partial y = x^3y + \frac{1}{2}x^2y^2 + g(x),$$

where $g(x)$ is some arbitrary function of x. Taking the partial derivative with respect to x, we then obtain

$$\psi_x = 3x^2y + xy^2 + g'(x),$$

and since $\psi_x = M = 3x^2y + xy^2$, this tells us

$$3x^2y + xy^2 + g'(x) = 3x^2y + xy^2.$$

Hence $g'(x) = 0$, meaning that we can take $g(x) = 0$ and $\psi(x, y) = x^3y + \frac{1}{2}x^2y^2$. Lastly, the solution is given by setting $\psi(x, y) = C$, ie:

$$x^3 y + \frac{1}{2} x^2 y^2 = C.$$

4. (10 points) Suppose a 120 gallon well-mixed tank initially contains 90 lb. of salt mixed with 90 gal. of water. Salt water (with a concentration of 2 lb/gal) comes into the tank at a rate of 4 gal/min. The solution flows out of the tank at a rate of 3 gal/min. How much salt is in the tank when it is full?

Solution. First of all, we notice that the volume of water V in the tank is nonconstant. In fact:

$$\frac{dV}{dt} = \overbrace{4}^{\text{rate in}} - \overbrace{3}^{\text{rate out}},$$

meaning that $dV/dt = 1$, and therefore $V = t + C$. Since the initial volume of water in the tank is 90 gallons, we get $C = 90$ and therefore $V = t + 90$ gallons. Since tha tank is full when $V = 120$, setting $t + 90 = 120$, we see that the tank will be full after 30 minutes.

Next, we determine the differential equation for the quantity S of salt in the tank:

$$\frac{dS}{dt} = \text{rate in} \quad - \quad \text{rate out},$$

where

$$\text{rate in} = \left(2\frac{\text{lbs}}{\text{gal}}\right) \cdot \left(4\frac{\text{gal}}{\text{min}}\right) = 8\text{lbs/min}.$$

and

$$\text{rate out} = \overbrace{\frac{S}{V}}^{\text{concentration (lbs/gal)}} \cdot \left(3\frac{\text{gal}}{\text{min}}\right) = 3S/V\text{lbs/min}.$$

and therefore recalling that $V = t + 90$:

$$\frac{dS}{dt} = 8 - 3\frac{S}{t+90}.$$

This is a linear equation; an integrating factor for this equation is $\mu(t) = (t + 90)^3$. Using this, we get the solution to be

$$S = 2(t + 90) + \frac{C}{(t + 90)^3},$$

and the initial condition $S(0) = 90$ tells us $C = -(90)^4$ so that

$$S = 2(t + 90) - \frac{(90)^4}{(t + 90)^3}.$$

Since the tank is full when $t = 30$, the quantity of salt in the tank is given by

$$S(30) = \approx 202.03 \text{ lbs.}$$

5. (a) (4 points) Find a particular solution of the equation

$$y'' + 2y' + 3y = \cos(t)$$

(b) (2 points) Find a particular solution of the equation

$$y'' + 2y' + 3y = \sin(t)$$

(c) (4 points) Find the general solution of the equation

$$y'' + 2y' + 3y = 3\cos(t) - 2\sin(t)$$

Solution.

(a) To find a particular solution to

$$y'' + 2y' + 3y = \cos(t),$$

we consider the "squigglyfied" equation

$$\widetilde{y}'' + 2\widetilde{y}' + 3\widetilde{y} = e^{it}.$$

Then since i is not a root of the characteristic polynomial $x^2 + 2x + 3$, we propose a particular solution of the form

$$\widetilde{y}_p = Ae^{it}.$$

We calculate $\widetilde{y}_p' = iAe^{it}$ and $\widetilde{y}_p'' = -Ae^{it}$, so that

$$\widetilde{y}_p'' + 2\widetilde{y}_p' + 3\widetilde{y}_p = 2(1+i)Ae^{it}.$$

Thus for $\widetilde{y}_p$ to be a particular solution of the squigglyfied equation, we need $2(1+i)Ae^{it} = e^{it}$ and therefore

$$A = \frac{1}{2(1+i)} = \frac{1}{4} - \frac{1}{4}i.$$

Hence

$$\widetilde{y}_p = \left(\frac{1}{4} - \frac{1}{4}i\right)e^{it} = \frac{1}{4}\cos(t) + \frac{1}{4}\sin(t) + i\left(-\frac{1}{4}\cos(t) + \frac{1}{4}\sin(t)\right)$$

Hence a particular solution is given byn

$$y_p = \mathrm{Re}(\widetilde{y}_p) = \frac{1}{4}\cos(t) + \frac{1}{4}\sin(t).$$

(b) Using the same $\widetilde{y}_p$ of (a), we immediately get a particular solution

$$y_p = \mathrm{Im}(\widetilde{y}_p) = -\frac{1}{4}\cos(t) + \frac{1}{4}\sin(t).$$

(c) Let y_1 be the particular solution found in (a) and y_2 be the particular solution found in (b). Then using the usual argument about linear operators, a particular solution y_p to the equation given in (c) is

$$y_p = 3y_1 - 2y_2 = \frac{5}{4}\cos(t) + \frac{1}{4}\sin(t).$$

The general solution of the corresponding homogeneous equation is $y_h = C_1 e^{-t}\cos(\sqrt{2}t) + C_2 e^{-t}\sin(\sqrt{2}t)$ and therefore the general solution of the equation in (c) must be

$$y = C_1 e^{-t}\cos(\sqrt{2}t) + C_2 e^{-t}\sin(\sqrt{2}t) + \frac{5}{4}\cos(t) + \frac{1}{4}\sin(t).$$

6. (10 points) A mass weighing 2 lb stretches a spring 16 ft. Suppose the mass is displaced an additional 1 ft downward and then released with an upward velocity of 2 ft/s. The mass is in a medium that exerts a viscous resistance of 2 lb when the mass has a velocity of 16 ft/s. Determine the quasi-amplitude and quasifrequency of the resultant motion.

Solution. From the question, we deduce that the spring constant is $k = 1/8$ lbs/ft, that the damping constant is $\gamma = 1/8$ lbs·s/ft, and that the mass is $m = 2/32 = 1/16$ lb·s^2/ft. Furthermore the length u of the mass-spring-system relative to its resting length satisfies the initial condition $u(0) = 1$ ft. and $u'(0) = -2$ ft/sec. Therefore the initial value problem we wish to solve is

$$\frac{1}{16}u'' + \frac{1}{8}u' + \frac{1}{8}u = 0, \quad u(0) = 1, \; u'(0) = -2.$$

The roots of the corresponding characteristic polynomial are $-1 \pm i$, so that the general solution to the corresponding homogeneous equation is

$$u(t) = C_1 e^{-t}\cos(t) + C_2 e^{-t}\sin(t).$$

Our initial conditions then say

$$1 = C_1$$

and also since

$$u'(t) = -C_1 e^{-t}\cos(t) - C_1 e^{-t}\sin(t) - C_2 e^{-t}\sin(t) + C_2 e^{-t}\cos(t)$$

$$-2 = -C_1 + C_2$$

so that since $C_1 = 1$, we must have $C_2 = -1$. Thus

$$u(t) = e^{-t}\cos(t) - e^{-t}\sin(t).$$

Now using our trig trick, we write

$$u(t) = Re^{-t}\cos(t - \delta)$$

for some value of δ and for

$$R = \sqrt{1^2 + (-1)^2} = \sqrt{2}.$$

Therefore the quasi-amplitude is $\sqrt{2}$ and the quasi-frequency is 1.

7. (10 points) Find the Laplace transform of

$$f(t) = \begin{cases} 0 & \text{if } 0 \leq t < 2\pi \\ \cos(t) + t & \text{if } 2\pi \leq t \end{cases}$$

Solution. We first convert $f(t)$ to step function form

$$f(t) = (\cos(t) + t)u_{2\pi}(t).$$

Now we wish to determine the laplace transform. To do so, we break it into the sum of two different functions

$$f(t) = \cos(t)u_{2\pi}(t) + tu_{2\pi}(t),$$

and find the Laplace transforms of each of the summands individually. First notice that since $\cos(t)$ is 2π periodic, $\cos(t) = \cos(t - 2\pi)$, and therefore

$$\cos(t)u_{2\pi}(t) = \cos(t - 2\pi)u_{2\pi}(t),$$

meaning that

$$\mathcal{L}\left\{\cos(t)u_{2\pi}(t)\right\} = \mathcal{L}\left\{\cos(t - 2\pi)u_{2\pi}(t)\right\} = e^{-2\pi s}\frac{s}{s^2 + 1}.$$

Furthermore

$$tu_{2\pi}(t) = (t - 2\pi + 2\pi)u_{2\pi}(t) = (t - 2\pi)u_{2\pi}(t) + 2\pi u_{2\pi}(t),$$

and therefore

$$\mathcal{L}\left\{tu_{2\pi}(t)\right\} = \mathcal{L}\left\{(t - 2\pi)u_{2\pi}(t)\right\} + \mathcal{L}\left\{2\pi u_{2\pi}(t)\right\} = e^{-2\pi s}\frac{1}{s^2} + e^{-2\pi s}\frac{2\pi}{s}.$$

Consequently

$$\mathcal{L}\{f(t)\} = e^{-2\pi s}\frac{s}{s^2 + 1} + e^{-2\pi s}\frac{1}{s^2} + e^{-2\pi s}\frac{2\pi}{s}.$$

8. (10 points) Find the inverse Laplace transform of

$$F(s) = \frac{3s-7}{(s+2)(s^2+2s+3)}$$

Solution. We first perform a partial fraction decomposition

$$\frac{3s-7}{(s+2)(s^2+2s+3)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+2s+3}$$

and by clearing denominators, we find

$$3s-7 = (s^2+2s+3)A + (s+2)(Bs+C).$$

Then plugging in $s=-2$, we get

$$-13 = 3A + 0(B(-2)+C),$$

so that $A = -13/3$. Also by comparing the leading coefficients, we get $0 = A+B$, so that $B = 13/3$. Finally, by plugging in $s=0$, we get $-7 = 3A+2C$, so that $C=3$. Therefore

$$F(s) = \frac{-13/3}{s+2} + \frac{(13/3)s+3}{s^2+2s+3}.$$

Now the inverse of the first term is easy:

$$\mathcal{L}^{-1}\left\{\frac{-13/3}{s+2}\right\} = \frac{-13}{3}e^{-2t}.$$

The inverse of the second term is a bit more complicated. To find it we complete the square and do a bit of algebraic manipulation

$$\frac{(13/3)s+3}{s^2+2s+3} = \frac{(13/3)s+3}{(s+1)^2+2} = \frac{13}{3}\frac{s+1}{(s+1)^2+2} + \frac{-4/3}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+2}.$$

Therefore

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{(13/3)s+3}{s^2+2s+3}\right\} &= \mathcal{L}^{-1}\left\{\frac{13}{3}\frac{s+1}{(s+1)^2+2}\right\} + \mathcal{L}^{-1}\left\{\frac{-4/3}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+2}\right\}\\ &= \frac{13}{3}e^{-t}\cos(\sqrt{2}t) + \frac{-4/3}{\sqrt{2}}e^{-t}\sin(\sqrt{2}t).\end{aligned}$$

Combining all of these together, we get

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{-13}{3}e^{-2t} + \frac{13}{3}e^{-t}\cos(\sqrt{2}t) + \frac{-4/3}{\sqrt{2}}e^{-t}\sin(\sqrt{2}t)$$

9. (10 points) Use Laplace transforms to find the solution to the initial value problem

$$y'' - 9y = e^t \sin(2t), \; y(0) = 1, \; y'(0) = -2.$$

Solution. We first calculate

$$\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0) = s^2\mathcal{L}(y) - s - 2$$

and also

$$\mathcal{L}(e^t \sin(2t)) = \frac{2}{(s-1)^2+4},$$

so that taking the Laplace transform of both sides of the original differential equation gives us

$$s^2\mathcal{L}(y) - s + 2 - 9\mathcal{L}(y) = \frac{2}{(s-1)^2+4}.$$

It follows that

$$\mathcal{L}(y) = \frac{s-2}{s^2-9} + \frac{2}{((s-1)^2+4)(s^2-9)}.$$

Now by partial fractions, we calculate

$$\frac{2}{((s-1)^2+4)(s^2-9)} = \frac{(-1/40)s - 1/8}{(s-1)^2+4} + \frac{1/24}{s-3} - \frac{1/60}{s+3}.$$

And also

$$\frac{s-2}{s^2-9} = \frac{5/6}{s+3} + \frac{1/6}{s-3}$$

so that

$$\mathcal{L}(y) = \frac{(-1/40)s - 1/8}{(s-1)^2+4} + \frac{5/24}{s-3} + \frac{49/60}{s+3}.$$

Lastly, notice that

$$\frac{(-1/40)s - 1/8}{(s-1)^2+4} = -\frac{1}{40}\frac{s-1}{(s-1)^2+4} - \frac{3}{40}\frac{2}{(s-1)^2+4}$$

and therefore

$$\mathcal{L}^{-1}\left(\frac{(-1/40)s - 1/8}{(s-1)^2+4}\right) = -\frac{1}{40}e^t\cos(2t) - \frac{3}{40}e^t\sin(2t)$$

and

$$\mathcal{L}^{-1}\left(\frac{5/24}{s-3}\right) = \frac{5}{24}e^{3t}$$

and

$$\mathcal{L}^{-1}\left(\frac{49/60}{s+3}\right) = \frac{49}{60}e^{-3t},$$

so that

$$y = -\frac{1}{40}e^t\cos(2t) - \frac{3}{40}e^t\sin(2t) + \frac{5}{24}e^{3t} + \frac{49}{60}e^{-3t}.$$

10. (10 points) Use Laplace transforms to find the solution to the initial value problem

$$y'' + 5y' + 6y = f(t), \quad y(0) = 0,\ y'(0) = 0,$$

where

$$f(t) = \begin{cases} 0 & \text{if } 0 \le t < \pi \\ 1 & \text{if } \pi \le t < 2\pi \\ 0 & \text{if } 2\pi \le t \end{cases}$$

Solution. We first convert $f(t)$ to step function form:

$$f(t) = u_\pi(t) - u_{2\pi}(t).$$

Therefore

$$\mathcal{L}\{f(t)\} = e^{-\pi s}\frac{1}{s} - e^{-2\pi s}\frac{1}{s}$$

and also

$$\mathcal{L}\{y'' + 5y' + 6y\} = (s^2 + 5s + 6)\mathcal{L}\{y\}.$$

so that

$$\mathcal{L}\{y\} = e^{-\pi s}\frac{1}{s(s^2 + 5s + 6)} - e^{-2\pi s}\frac{1}{s(s^2 + 5s + 6)}.$$

Now we perform a partial fraction decomposition

$$\frac{1}{s(s^2 + 5s + 6)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3},$$

and then the usual means allow us to deduce $A = 1/6, B = -1/2, C = 1/3$ and therefore

$$\frac{1}{s(s^2 + 5s + 6)} = \frac{1/6}{s} + \frac{-1/2}{s+2} + \frac{1/3}{s+3},$$

from which we conclude

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2 + 5s + 6)}\right\} = \frac{1}{6} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t}.$$

It follows that

$$\mathcal{L}^{-1}\left\{e^{-\pi s}\frac{1}{s(s^2 + 5s + 6)}\right\} = \left(\frac{1}{6} - \frac{1}{2}e^{-2(t-\pi)} + \frac{1}{3}e^{-3(t-\pi)}\right)u_\pi(t).$$

and also that

$$\mathcal{L}^{-1}\left\{e^{-2\pi s}\frac{1}{s(s^2 + 5s + 6)}\right\} = \left(\frac{1}{6} - \frac{1}{2}e^{-2(t-2\pi)} + \frac{1}{3}e^{-3(t-2\pi)}\right)u_{2\pi}(t).$$

Thus

$$y = \left(\frac{1}{6} - \frac{1}{2}e^{-2(t-\pi)} + \frac{1}{3}e^{-3(t-\pi)}\right)u_\pi(t) - \left(\frac{1}{6} - \frac{1}{2}e^{-2(t-2\pi)} + \frac{1}{3}e^{-3(t-2\pi)}\right)u_{2\pi}(t).$$

Figure 1: Elementary Laplace Transforms:

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$1/s$
e^{at}	$\frac{1}{s-a}$
t^n, $n \geq 0$ integer	$\frac{n!}{s^{n+1}}$
t^p, $p \geq 0$ real	$\frac{\Gamma(p+1)}{s^{p+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$t^n e^{at}$ $n \geq 0$ integer	$\frac{n!}{(s-a)^{n+1}}$
$u_c(t)$	$\frac{e^{-cs}}{s}$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{ct}f(t)$	$F(s-c)$
$f(ct)$	$\frac{1}{c}F(s/c)$
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
$\delta(t-c)$	e^{-cs}
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
$(-t)^n f(t)$	$F^{(n)}(s)$