Math 307 E - Summer 2011
Pactice Mid-Term Exam
June 18, 2011

Name: ______________________________ Student number: ____________________

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	60	

- Complete all questions.
- You may use a scientific calculator during this examination. Other electronic devices (e.g. cell phones) are not allowed, and should be turned off for the duration of the exam.
- You may use one hand-written 8.5 by 11 inch page of notes.
- Show all work for full credit.
- You have 60 minutes to complete the exam.

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1. (a) Solve for $y(t)$:

$$y' = 3(t+y). \qquad y(0) = y_0.$$

(5 points)

The equation is linear: $y' - 3y = 3t$. The integrating factor $\mu(t) = e^{\int -3dt} = e^{-3t}$. Then we multiply by the integrating factor to get

$$e^{-3t}y' - 3e^{-3t}y = \frac{d}{dt}\left(e^{-3t}y\right) = 3te^{-3t}.$$

Integrating, we have $e^{-3t}y = \int 3te^{-3t} = \frac{1}{3}\int ue^{-u}du = \frac{1}{3}(-ue^{-u} - e^{-u}) + C = \frac{-1}{3}(3t+1)e^{-3t} + C$, where $u = 3t$. Then

$$y = -t - \frac{1}{3} + Ce^{-3t}.$$

(b) Find the general solution to

$$(\cos^2 x)y' = y^2 - 1$$

(5 points)

The equation is separable. We get $\frac{dy}{y^2-1} = \sec^2(x)dx$. To integrate the left hand side, we use partial fractions:

$$\frac{1}{y^2-1} = \frac{1}{(y-1)(y+1)} = \frac{1/2}{y-1} + \frac{-1/2}{y+1},$$

(use the cover-up method since we have distinct linear factors). Then integrating we get

$$\frac{1}{2}(\ln(y-1) - \ln(y+1)) = \tan(x) + C.$$

This is sufficient for full credit, though we can solve for y. Observe:

$$\begin{aligned}
\ln\left(\frac{y-1}{y+1}\right) &= 2\tan(x) + C_2 \\
\left(\frac{y-1}{y+1}\right) &= C_3e^{2\tan(x)} \\
1 - \frac{2}{y+1} &= C_3e^{2\tan(x)} \\
\frac{1}{1-C_3e^{2\tan(x)}} &= \frac{y+1}{2} \\
\frac{2}{1-C_3e^{2\tan(x)}} - 1 &= y.
\end{aligned}$$

2. (a) Find the general solution to

$$xy\frac{dy}{dx} = (y-x)(y+x), \quad x, y > 0.$$

(5 points)

This equation is homogeneous. Indeed:

$$\frac{dy}{dx} = \frac{y^2 - x^2}{xy} = \frac{y}{x} - \frac{x}{y};$$

letting $ux = y$, so that $u + x\frac{du}{dx} = \frac{dy}{dx}$, we get

$$u + x\frac{du}{dx} = u - \frac{1}{u},$$

or canceling and separating:

$$udu = -\frac{1}{x}dx.$$

We integrate to find that $\frac{u^2}{2} = -\ln(x) + C$; substituting back $u = \frac{y}{x}$ we get

$$y^2 = -2x^2\ln(x) + C_2x^2.$$

(b) Find the general solution to

$$1 + \frac{1}{y\sqrt{x}} - \frac{2\sqrt{x}}{y^2}\frac{dy}{dx} = 0, \quad x, y > 0.$$

(5 points) This equation is exact. Indeed, $M(x,y) = 1 + \frac{1}{y\sqrt{x}}$, $N(x,y) = -\frac{2\sqrt{x}}{y^2}$, and

$$M_y = -\frac{1}{y^2\sqrt{x}} = N_x.$$

Thus, there exists a potential function $\phi(x,y)$, and $\phi = \int Mdx + h(y)$. We check that

$$\int Mdx = x + \frac{2\sqrt{x}}{y},$$

so that $\phi = x + \frac{2\sqrt{x}}{y} + h(y)$. Then $\phi_y = -2\frac{\sqrt{x}}{y^2} + h'(y) = N(x,y) = -\frac{2\sqrt{x}}{y^2}$. Thus $h'(y) = 0$, and so $h(y)$ is an arbitrary constant (which we choose to be zero). Then the differential equation is

$$\frac{d}{dx}\left(\phi(x,y)\right) = 0,$$

so the solution is (given implicitly by) $\phi(x,y) = x + 2\frac{\sqrt{x}}{y^2} = C$.

3. (a) We consider a primary fermentation tank in some brewery. There is a population of yeast in the tank, which grows logistically to a carrying capacity of 10 in the absence of alcohol. When the population of the yeast is small, the (unrestricted) growth rate is $r = 2$. Write down (but do not solve) a differential equation which models $P(t)$, the population of yeast in the tank at time t. (4 points)

$$\frac{dP(t)}{dt} = 2P(t)\left(1 - \frac{P(t)}{10}\right).$$

(b) The yeast consume sugar and produce alcohol, which is toxic to them. Assume that yeast cells die at a rate proportional to the product of the amount of yeast, $P(t)$, and the amount $A(t)$ of alcohol at time t. Call this proportionality constant α. Modify the previous differential equation to account for this phenomenon (but do not solve it). (4 points)

$$\frac{dP(t)}{dt} = 2P(t)\left(1 - \frac{P(t)}{10}\right) - \alpha\, P(t)\, A(t).$$

(c) The rate of change of alcohol, $A'(t)$, is directly proportional to the population of yeast at that time. Call this proportionality constant β. Write this down as a differential equation (but do not solve it). (Note: These last two differential equations form a *system* of differential equations which model the population of yeast in the tank.) (2 points)

$$\frac{dA(t)}{dt} = \beta\, P(t).$$

4. A circus act involves shooting a man of 80 kg from a cannon straight into the air from some platform h meters above the ground.

   (a) Assuming the initial velocity is 50 m/s directly upward, find the time until the man begins to fall. Neglect air resistance. Assume $g = 10m/s^2$ for simplicity. (5 points)

   Let $v(t)$ be the velocity at time t. Then $v'(t)$ is the acceleration, and since $F = ma = -mg$, we get $v'(t) = -g$, so that $v(t) = -gt + C$, where $C = v(0)$. Since the velocity of the object at time $t = 0$ is 50, we get $v(t) = -gt + 50$. Since the velocity of the man when he begins to fall is 0, we just solve $0 = -gt + 50$ for t, so $t = 5$ sec.

   (b) 1 second after the man begins to fall from his maximum altitude, he opens his parachute; his parachute provides a force of air resistance of $10|v(t)|$, where $v(t)$ is the velocity of the man at time t.

      i. Set up, but do not solve, the initial value problem modeling the man's velocity as a function of time during this period. Your initial conditions will involve h.
      ii. Find any equilibrium solutions to that problem, and classify as stable, unstable, or semi-stable.

   (5 points)

      i. Here, we have $F = ma = -mg - 10v$, since $v \leq 0$ during this interval of time. Then the differential equation is

$$\frac{dv}{dt} = -g - \frac{10}{80}v = -10 - \frac{1}{8}v, \qquad v(6) = -g(6) + 50 = -10.$$

      ii. Equilibrium solutions occur when $-10 - \frac{1}{8}v = \frac{dv}{dt} = 0$, i.e. when $v = -80$. Graphing the function $-10 - \frac{1}{8}v$ shows that this is a stable equilibrium.

5. Solve the following second-order differential equations:

(a) $y'' - 2y' - 3y = 0$. (3 points)

The characteristic equation $r^2 - 2r - 3 = 0 = (r-3)(r+1)$, so $r = -1, 3$. Thus the two fundamental solutions are e^{-t}, and e^{3t}; the general solution is then

$$y(t) = c_1 e^{-t} + c_2 e^{3t}.$$

(b) $4y'' + 4y' + y = 0$. (3 points)

The char. eq. is $4r^2 + 4r + 1 = 0 = 4(r + \frac{1}{2})^2$, so $r = -1/2$ with multiplicty 2. Thus the two fundamental solutions are $e^{-1/2t}$ and $te^{-1/2t}$; the general solution is then

$$y(t) = c_1 e^{-t/2} + c_2 t e^{-t/2}.$$

(c) $\pi y'' + \gamma y' + ey = 0$, where $\gamma \approx 0.57721566$ is called the Euler-Mascheroni constant. (4 points)

The char. eq. is $\pi r^2 + \gamma r + e = 0$, so $r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\pi\, e}}{2\pi}$; these are complex numbers, since $\gamma^2 < 1 < 4\pi\, e$. Thus $r = -\frac{\gamma}{2\pi} \pm \frac{\sqrt{4\pi\, e - \gamma^2}}{2\pi} i$. Choosing the positive root, we get

$$e^{rt} = e^{-\gamma t/(2\pi)} \cdot (\cos(\frac{\sqrt{4\pi\, e - \gamma^2}}{2\pi} t) + \sin(\frac{\sqrt{4\pi\, e - \gamma^2}}{2\pi} t)).$$

Taking real and imaginary parts, we get:

$$y_1 = e^{-\gamma t/(2\pi)} \cos(\frac{\sqrt{4\pi\, e - \gamma^2}}{2\pi} t) \qquad y_2 = e^{-\gamma t/(2\pi)} \sin(\frac{\sqrt{4\pi\, e - \gamma^2}}{2\pi} t).$$

Of course, the general solution is a linear combination

$$y(t) = c_1 y_1 + c_2 y_2.$$

6. Assume that $y_1(t) = t$ is a solution to the differential equation

$$y'' + \left(1 - \frac{2}{t}\right) y' - \frac{t-2}{t^2} y = 0 \qquad t > 0$$

(this is easy to check if you want). Find another independent solution using the method of reduction of order, and write down the general solution to the differential equation. (10 points)

We guess that the second solution $y_2(t) = w(t) \cdot y_1(t) = t \cdot w$. Then $y_2' = w + tw'$, and $y_2'' = 2w' + tw''$. Plugging this into the differential equation, we get

$$\begin{aligned} 0 &= [2w' + tw''] + \left(1 - \frac{2}{t}\right)[w + tw'] - \frac{t-2}{t^2}[tw] \\ &= tw'' + (2 + t - 2)w' + \left(1 - \frac{2}{t} - \frac{t^2 - 2t}{t^2}\right) w. \end{aligned}$$

Notice that the coefficient of w is actually 0 (as it must be since $y_1(t) = t$ is a solution to this linear equation – always expect this!). Then let $v = w'$; the equation reduces to

$$v' = -v,$$

which we can easily solve to obtain $v(t) = e^{-t}$. Then $w = \int v dt = -e^{-t}$. It follows that $y_2 = w(t)y_1(t) = -e^{-t} \cdot t$ is a solution. It follows that te^{-t} is also a solution, since we can multiply any solution by -1 and still have a solution. Then the general solution is

$$y(t) = c_1 t + c_2 t e^{-t}.$$