1. (4 points) Find the solution to the initial value problem. Simplify your answer.

$$y' = te^{t+y}, \quad y(0) = 0$$

Since $e^{t+y} = e^t e^y$, the equation is separable with

$$y'e^{-y} = te^t.$$

Integrate both sides with respect to $t$ and change variables $u = y(t)$ in the right integral, so $\mathrm{d}u = y'(t)\mathrm{d}t$, to obtain

$$-e^{-y(t)} = -e^{-u} = \int e^{-u}\mathrm{d}u = \int e^{-y(t)}y'(t)\mathrm{d}t = \int te^t\mathrm{d}t$$

To integrate $te^t$ use integration by parts,

$$\int te^t\mathrm{d}t = te^t - \int e^t\mathrm{d}t = te^t - e^t + C.$$

Thus

$$-e^{-y(t)} = te^t - e^t + C.$$

To determine $C$, evaluate this equation at $t = 0$ and use the initial condition:

$$-1 = -e^0 = -e^{-y(0)} = te^t - e^t + C\Big|_{t=0} = -1 + C \quad \Rightarrow \quad C = 0.$$

Solving for $y(t)$ yields

$$\begin{aligned} & -e^{-y(t)} = te^t - e^t \\ \Rightarrow \quad & e^{-y(t)} = e^t - te^t \\ \Rightarrow \quad & -y(t) = \ln(e^t - te^t) \\ \Rightarrow \quad & y(t) = -\ln(e^t - te^t) \end{aligned}$$

This can be simplified using the log rules to

$$y(t) = -\ln(e^t - te^t) = -\ln((1-t)e^t) = -\ln(1-t) - \ln(e^t) = -t - \ln(1-t).$$

2. (6 total points)

   (a) (5 points) Find all the solutions to the differential equation

$$y' + 2ty + t = 0$$

The equation is linear, so we use integrating factors:

$$\mu y' + \mu 2ty = -\mu t,$$

so $\mu$ needs to satisfy $\mu' = 2t\mu$, a separable differential equation. Now,

$$\ln|\mu| = \int \frac{\mu'(t)}{\mu(t)}\mathrm{d}t = \int 2t\mathrm{d}t = t^2,$$

so one solution is $\mu = e^{t^2}$. Thus

$$\left(e^{t^2} y\right)' = e^{t^2} y' + 2te^{t^2} y = -te^{t^2}.$$

Use the substitution $u = t^2$ (so $\mathrm{d}u = 2t\mathrm{d}t$) to integrate both sides,

$$e^{t^2} y = \int -te^{t^2} \mathrm{d}t = -\frac{1}{2}\int e^u \mathrm{d}u = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{t^2} + C.$$

We conclude that all the solutions are

$$y = -\frac{1}{2} + Ce^{-t^2}.$$

(b) (1 point) Determine an initial condition such that the solution to differential equation does not grow or decay exponentially.

If the solution may not decay or grow exponentially, we need $C = 0$. Evaluate the solution at $t = 0$ to obtain

$$y(0) = -\frac{1}{2} + C \quad \Rightarrow \quad C = y(0) + \frac{1}{2}$$

If we want $C = 0$, we need to choose the initial condition $y(0) = -\frac{1}{2}$.

3. (4 total points) Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = e^y - 1, \quad -\infty < y_0 < \infty$$

where $y(0) = y_0$.

(a) (2 points) Determine and classify all equilibrium solutions.

Equilibrium solutions satisfy $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$, so

$$0 = \frac{\mathrm{d}y}{\mathrm{d}t} = e^y - 1 \quad \Leftrightarrow \quad e^y = 1 \quad \Leftrightarrow \quad y = \ln(1) = 0,$$

so $y = 0$ is the only equilibrium solution. To classify it, note that if $y > 0$, then $e^y > 1$, and therefore $\frac{\mathrm{d}y}{\mathrm{d}t} = e^y - 1 > 0$. Conversely, if $y < 0$, then $e^y < 1$, and $\frac{\mathrm{d}y}{\mathrm{d}t} = e^y - 1 < 0$. Conclude that $y = 0$ is asymptotically unstable.

(b) (2 points) Determine for which values of $y$ the function is concave and convex, respectively.

First, compute the second derivative using the chain rule,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}e^y - 1 = \frac{\mathrm{d}}{\mathrm{d}y}(e^y - 1) \cdot \frac{\mathrm{d}y}{\mathrm{d}t} = e^y \cdot (e^y - 1).$$

Since always $e^y > 0$, this equation has only one zero, $y = 0$; and the analysis from (a) shows that if $y > 0$, then $\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} > 0$, and if $y < 0$, then $\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} < 0$. Therefore, $y$ is convex for on $(0, \infty)$ and concave on $(-\infty, 0)$.

4. (3 total points) A mass of 1kg is thrown into the air vertically (i. e. wither upward or downward). The initial velocity and initial position are not specified. The air resistance is measured to be $\gamma|v|$, where $v$ is the velocity of the mass and $\gamma$ is some positive constant that is to be determined. Take the acceleration due to gravity to be 10m/s$^2$.

   (a) (2 points) Write down a differential equation that describes the velocity. Your answer should involve $\gamma$ as a constant.

   Suppose that upward is the positive direction. Then by Newton's law,

   $$mv' = F = -mg - \gamma v.$$

   Since $m = 1$ and $g = 10$ by assumption, we obtain

   $$v' = F = -10 - \gamma v.$$

   (b) (1 point) Suppose it is known that the velocity of the mass will remain constant for the whole duration of the motion if the mass is thrown downward at an initial speed of 100 m/s. Find $\gamma$.

   Since the velocity is constant, $v(t) = v(0) = -100$ (remember that the upward direction was chosen to be positive) for all times $t$, and $v' = 0$. Therefore,

   $$0 = v' = -10 - \gamma v = -10 - \gamma \cdot (-100) \quad \Rightarrow \quad \gamma = \frac{1}{10}.$$

5. (7 points) The population of mosquitoes in a certain area increases at a rate proportional to the current population, and in absence of other factors, the population doubles each week. There are 200,000 mosquitoes in the area initially, and predators (birds, bats, and so forth) eat 20,000 mosquitoes/day. Determine the population of mosquitoes in the area at any time.

   It is important to carefully read the problem: The model for the number of mosquitoes is given by

   $$\frac{\mathrm{d}P}{\mathrm{d}t} = kP - 20,000, \quad P(0) = 200,000$$

   where $P(t)$ is the number of mosquitoes on day $t$. Yet, we are told that the number of mosquitoes doubles every week if there are no predators, so this information had to be used on the model

   $$\frac{\mathrm{d}P^*}{\mathrm{d}t} = kP^*.$$

   The fact that the mosquitoes double each week under this model means $P^*(t+7) = 2P^*(t)$. We first solve the second equation and use the given information to determine $k$, and then use this to solve the first equation.

   The second equation is separable, so if we set $u = P^*(t)$, so $\mathrm{d}u = (P^*(t))'\mathrm{d}t$, and

   $$\ln(P^*(t)) = \ln|u| = \int \frac{1}{u}\mathrm{d}u = \int \frac{(P^*(t))'}{P^*(t)}\mathrm{d}t = \int k\mathrm{d}t = kt + C.$$

   Thus

   $$P^*(t) = Ce^{kt}.$$

Since $P^*(t+7) = 2P^*(t)$, we conclude that

$$\begin{aligned} & Ce^{k(t+7)} = P^*(t+7) = 2P^*(t) = 2Ce^{kt} \\ \Rightarrow\ & Ce^{kt}e^{7k} = 2Ce^{kt} \\ \Rightarrow\ & e^{7k} = 2 \\ \Rightarrow\ & k = \frac{\ln(2)}{7}. \end{aligned}$$

Thus our initial model reads

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\ln(2)}{7}P - 20{,}000, \quad P(0) = 200{,}000.$$

This equation is again separable. Using the usual change of variables, $u = P(t)$, so $\mathrm{d}u = P'(t)\mathrm{d}t$,

$$\begin{aligned} \int \frac{P'(t)}{\frac{\ln(2)}{7}P(t) - 20{,}000}\mathrm{d}t &= \int \frac{1}{\frac{\ln(2)}{7}u - 20{,}000}\mathrm{d}u \\ &= \frac{7}{\ln(2)} \ln\left(\frac{\ln(2)}{7}u - 20{,}000\right) \\ &= \frac{7}{\ln(2)} \ln\left(\frac{\ln(2)}{7}P(t) - 20{,}000\right) \end{aligned}$$

Thus

$$\frac{7}{\ln(2)} \ln\left(\frac{\ln(2)}{7}P(t) - 20{,}000\right) = \int \frac{P'(t)}{\frac{\ln(2)}{7}P(t) - 20{,}000}\mathrm{d}t = \int 1\mathrm{d}t = t + C.$$

Solving for $P(t)$ yields

$$\begin{aligned} & \frac{7}{\ln(2)} \ln\left(\frac{\ln(2)}{7}P(t) - 20{,}000\right) = t + C \\ \Rightarrow\ & \ln\left(\frac{\ln(2)}{7}P(t) - 20{,}000\right) = \frac{\ln(2)}{7}t + C \\ \Rightarrow\ & \frac{\ln(2)}{7}P(t) - 20{,}000 = Ce^{\frac{\ln(2)}{7}t} \\ \Rightarrow\ & P(t) = Ce^{\frac{\ln(2)}{7}t} + \frac{140{,}000}{\ln(2)} \end{aligned}$$

Now use the initial condition to determine $C$,

$$200{,}000 = P(0) = C + \frac{140{,}000}{\ln(2)} \quad \Rightarrow \quad C = 200{,}000 - \frac{140{,}000}{\ln(2)}.$$

Thus

$$P(t) = \left(200{,}000 - \frac{140{,}000}{\ln(2)}\right) e^{\frac{\ln(2)}{7}t} + \frac{140{,}000}{\ln(2)}$$

Since $e^{\frac{\ln(2)}{7}t} = e^{\ln\left(2^{\frac{t}{7}}\right)} = 2^{\frac{t}{7}}$, this can be simplified to

$$P(t) = \left(200{,}000 - \frac{140{,}000}{\ln(2)}\right) 2^{\frac{t}{7}} + \frac{140{,}000}{\ln(2)}.$$