Your Name

Student ID #

1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	

- Complete all questions. BOX your answers. Do not write outside the marginal lines.
- One handwritten two-sided sheet of note and calculator are allowed. NO CHEATING!
- In order to receive credit, you must **show all of your work**; to obtain full credit, you must **provide mathematical justifications**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Raise your hand if you have a question.
- You have 50 minutes to complete the midterm.

$\int x^a\,\mathrm{d}x = \frac{x^{a+1}}{a+1}$

$\int e^x\,\mathrm{d}x = e^x$

$\int \sin x\,\mathrm{d}x = -\cos x$

$\int \sec^2 x\,\mathrm{d}x = \tan x$

$\int \csc x \cot x\,\mathrm{d}x = -\csc x$

$\int \sec x\,\mathrm{d}x = \ln|\sec x + \tan x|$

$\int \tan x\,\mathrm{d}x = \ln(\sec x)$

$\int \sinh x\,\mathrm{d}x = \cosh x$

$\int \frac{\mathrm{d}x}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$

$\int \frac{\mathrm{d}x}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$

$\int \frac{1}{x}\,\mathrm{d}x = \ln|x|$

$\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a}$

$\int \cos x\,\mathrm{d}x = \sin x$

$\int \sec x \tan x\,\mathrm{d}x = \sec x$

$\int \csc^2 x\,\mathrm{d}x = -\cot x$

$\int \csc x\,\mathrm{d}x = \ln|\csc x + \cot x|$

$\int \cot x\,\mathrm{d}x = \ln(\sin x)$

$\int \cosh x\,\mathrm{d}x = \sinh x$

$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right)$

$\int \frac{\mathrm{d}x}{\sqrt{x^2\pm a^2}} = \ln\left|x \pm \sqrt{x^2+a^2}\right|$

1.a (4points) Find a differential equation whose general solution is $y = c_1 e^{-2t}\cos(2t) + c_2 e^{-2t}\sin(2t)$.

*S*olution. Two roots of the characteristic equation are $r = -2 \pm 2i$. So $(r+2)^2 = (2i)^2 = -4$. This implies

$$r^2 + 4r + 8 = 0.$$

So the differential equation is $y'' + 4y' + 8y = 0$.

1.b.(3 pts) Find a differential equation whose general solution is $y = c_1 e^{-2t}\cos(2t) + c_2 e^{-2t}\sin(2t) + \sin(2t)$.

From part a, the differential equation is

$$y'' + 4y' + 8y = (\sin 2t)'' + 4(\sin 2t)' + 8\sin 2t = 4\sin 2t + 8\sin 2t.$$

1.c. (3pts) Find a differential equation whose general solution is $y = c_1 e^{-2t} + c_2 t e^{-2t}$.
The characteristic equation has $r = 2$ as double root. So it is

$$(r+2)^2 = 0.$$

It follows that $y'' + 4y' + 4y = 0$.

2. (10 points) Solve the following initial value problem:

$$y'' + 2y' + 2y = (5t-1)e^t + 3, \quad y(0) = y'(0) = 1.$$

*S*olution. The characteristic equation $r^2 + 2r + 2 = 0$ has two roots $r = -1 \pm i$.
The general solution of the homogeneous equation is

$$y_h = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t.$$

The suitable form for the particular solution is $y_p = (At+B)e^t + C$.
After some calculation, we see that

$$y_p'' + 2y_p' + 2y_p = 5Ate^t + (4A+5B)e^t + 2C.$$

It implies that $A = 1, B = -1, C = 3/2$. The general solution of the original equation is

$$y = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t + te^t - e^t + 3/2.$$

Using the initial values, we get $c_1 = 1/2$ and $c_2 = 3/2$. So the answer is

$$y = \frac{1}{2}e^{-t}\cos t + \frac{3}{2}e^{-t}\sin t + te^t - e^t + \frac{3}{2}.$$

3. (10 pts) A mass that weighs 8 lb stretches a spring 6 in. The system is acted on by an external force of $8\sin(8t)$ lb. If the mass is pulled down 3 in and then released, determine the position of the mass at any time. There is no damped force.

*S*olution. $m = \frac{8}{32} = \frac{1}{4}$.
$k = \frac{8}{1/2} = 16$. $\gamma = 0$.
The equation is

$$\frac{1}{4}u'' + 16u = 8\sin 8t$$

or

$$u'' + 64u = 32\sin 8t.$$

The initial condition is $u(0) = \frac{1}{4}$, $u'(0) = 0$.
The characteristic equation has roots $r = \pm 8i$. So $u_g = c_1\cos 8t + c_2\sin 8t$.

The particular solution is of the form $u_p = t(A\cos 8t + B\sin 8t)$. Solve for A and B we get $A = 0, B = -2$. Therefore

$$u = u_g + u_p = c_1\cos 8t + c_2\sin 8t - 2t\cos 8t.$$

Using the initial values, we get

$$c_1 = \frac{1}{4} \text{ and } 8c_2 - 2 = 0.$$

So

$$u = \frac{1}{4}\cos 8t + \frac{1}{4}\sin 8t - 2t\cos 8t.$$

4. (10 pts) Given $y_1(t) = t$ satisfying the following differential equation, find a second solution of this equation:

$$t^2 y'' - t(t+2)y' + (t+2)y = 0, \;\; t > 0.$$

*S*olution. We use the method of reduction of order. Suppose $y = vy_1$ and we want to find v. By using the formula in class, we have

$$t^2 v'' y_1 + (2t^2 y_1' - t(t+2)y_1)v' = 0,$$
$$t^3(v'' - v') = 0 \text{ or } v'' - v' = 0.$$

This differential equation implies $v = Ce^t + D$. So

$$y = vy_1 = Cte^t + Dt.$$

We choose a second solution $y_2 = te^t$.

5. (10 pts) Find the general solution of the following differential equation:

$$t^2y'' - t(t+2)y' + (t+2)y = t^3 \sin t, \quad t > 0.$$

*S*olution. From problem 4, the general solution of the homogeneous equation is

$$y_h = c_1te^t + c_2t.$$

We need to find a particular solution. There are several ways to "guess". One way is the same method as in problem 4. Suppose the particular solution is $y_p = vy_1$ where $y_1 =$ and we want to find v.
Note that y_1 is a solution of the homogeneous equation, by using the same formula in class, the left hand side of the original equation is

$$t^3(v'' - v').$$

This implies that

$$t^3(v'' - v') = t^3 \sin t.$$

So $v'' - v' = \sin t$. Solve this differential equation,

$$v = Ce^t + \frac{1}{2}(\cos t - \sin t).$$

So $y_p = vt = Cte^t + \frac{1}{2}(\cos t - \sin t)t$. We choose

$$y_p = \frac{1}{2}(\cos t - \sin t)t.$$

The general solution of the original equation is

$$y = c_1te^t + c_2t + \frac{1}{2}(\cos t - \sin t)t.$$