# MATH 307D
# Midterm 2 Solution
# August 2, 2013

Name ______________________________ Student ID #______________

- Your exam should consist of this cover sheet, followed by 5 problems. Check that you have a complete exam.
- Unless otherwise indicated, show all your work and justify your answers.
- Unless otherwise indicated, your answers should be exact values rather than decimal approximations. For example, $\frac{\pi}{4}$ is an exact answer and is preferable to 0.7854.
- You may use a scientific calculator and one double-sided 8.5×11-inch sheet of handwritten notes. All other electronic devices, including graphing or programmable calculators, and calculators which can do calculus, are forbidden.
- The use of headphones, earbuds during the exam is not permitted. Turn off all your electronic devices and put them away.
- If you need more space, write on the back and indicate this. If you still need more space, raise your hand and I'll give you some extra paper to staple onto the back of your test.
- Academic misconduct will guarantee a score of zero on this exam. DO NOT CHEAT.

| Problem | Points | S C O R E |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| Total: | 50 | |

1. (10 points) Find the solution to the initial value problem

$$y'' - 10y' + 25y = 0, \quad y(0) = 1, \; y'(0) = 1.$$

**Solution:** The characteristic equation is

$$\lambda^2 - 10\lambda + 25 = 0,$$

with the double root $\lambda = 5$. Therefore, the general solution is

$$y(t) = C_1 t e^{5t} + C_2 e^{5t}.$$

Let us find the constants. $y(0) = C_2 = 1$, so and $y'(t) = C_1 e^{5t} + 5C_1 t e^{5t} + 5C_2 e^{5t}$, so $y'(0) = C_1 + 5C_2 = 1$. Therefore, $C_1 = 1 - 5C_2 = -4$, and

$$\boxed{-4te^{5t} + e^{5t}}$$

2. (10 points) Find the general solution to the following equation

$$y'' - 6y' + 8y = (2t + 3)e^t + 3te^{4t} + t^2 + \cos t.$$

You do NOT need to find coefficients for the particular solution. An example:

$$y'' + y = t \quad \Rightarrow \quad y = C_1 \cos t + C_2 \sin t + At + B,$$

where $C_1, C_2$ are arbitrary real-valued constants, and $A, B$ are coefficients to be determined by plugging into the equation. Your answer should be in a similar form.

**Solution:** The chatacteristic equation is

$$\lambda^2 - 6\lambda + 8 = 0 \ \Rightarrow \ \lambda = 2, 4.$$

The general solution of the homogeneous equation is

$$C_1 e^{2t} + C_2 e^{4t}.$$

A particular solution to the nonhomogeneous equation

$$y'' - 6y' + 8y = (2t + 3)e^t$$

is $y = (A_1 t + B_1)e^t$. Indeed, $\lambda = 1$ is not a root of the characteristic equation, so we do not need to raise the degree of the polynomial multiplied by $e^t$. It was linear and it will remain linear.

A particular solution to the nonhomogeneous equation

$$y'' - 6y' + 8y = 3te^{4t}$$

is $y(t) = (A_2 t^2 + B_2 t + C_2)e^{4t}$. Indeed, $\lambda = 4$ is a root of the characteristic equation, so we need to raise the degree of polynomial in the right-hand side by one. It was linear and will be quadratic.

Similarly,

$$y'' - 6y' + 8y = t^2 \quad \Rightarrow \quad y = A_3 t^2 + B_3 t + C_3.$$

$$y'' - 6y' + 8y = \cos t \quad \Rightarrow \quad y = A_4 \cos t + B_4 \sin t.$$

The general solution of the original equation is

$$\boxed{y = C_1 e^{2t} + C_2 e^{4t} + (A_1 t + B_1)e^t + (A_2 t^2 + B_2 t + C_2)e^{4t} + A_3 t^2 + B_3 t + C_3 + A_4 \cos t + B_4 \sin t}$$

Here, $C_1, C_2$ are arbitrary real constants, and $A_i, B_i, C_i$ are coefficients which are to be determined.

3. (10 points) Solve the equation

$$t^2 y'' - 8ty' + 8y = t^2$$

using variation of parameters. One solution is given: $y_1(t) = t$.

**Solution:** Any function $y(t) = Ct$ is also a solution to this equation. Let $C = C(t)$. Then

$$y(t) = C(t)t,\ y'(t) = C't + C,\ y''(t) = C''t + 2C'.$$

Then we get: $tC'' - 6C' = 1$. Let $z = C'$. Then $tz' - 6z = 1$. Solve the homogeneous equation $tz' - 6z = 0$:

$$\frac{dz}{z} = \frac{6dt}{t} \quad \Rightarrow \quad \log|z| = 6\log|t| + C_1 \quad \Rightarrow \quad |z| = e^{C_1}|t|^6 \quad \Rightarrow \quad z = \pm e^{C_1}t^6.$$

The constant $C_2 = \pm e^{C_1}$ can assume any nonzero values. But we lost the solution $z = 0$, which can be incorporated into this general formula by letting $C_2 = 0$. So $z = C_2 t^6$ is the general solution to the equation $tz' - 6z = 0$. And to find the solution of $tz' - 6z = 1$, use the method of variation of parameters again! Let $z = C_2(t)t^6$, then $z' = 6t^5 C_2(t) + C_2'(t)t^6$, so we have, after simplifying:

$$t^7 C_2' = 1 \quad \Rightarrow C_2 = t^{-7} \quad \Rightarrow \quad C_2 = -\frac{1}{6}t^{-6} + C_3.$$

Therefore,

$$z = C_2 t^6 = -\frac{1}{6} + C_3 t^6, \quad C = \int z dt = -\frac{t}{6} + C_3 t^7 + C_4.$$

Finally,

$$y = Ct = \boxed{-\frac{1}{6}t^2 + C_3 t^8 + C_4 t}$$

4. (10 points) Consider a spring with a ball of mass $m = 1$, with damping coefficient $\gamma = 2$, and spring constant $k = 4$. Suppose it starts from $u(0) = 1$, with velocity $u'(0) = 0$. Find $u(t)$.

**Solution:** We have:

$$u'' + 2u' + 4u = 0, \quad u(0) = 1, \ u'(0) = 0.$$

The characteristic equation:

$$\lambda^2 + 2\lambda + 4 = 0 \quad \Rightarrow \quad \lambda_{1,2} = -1 \pm \sqrt{3}i.$$

The general solution is

$$u(t) = C_1 e^{-t} \cos\left(\sqrt{3}t\right) + C_2 e^{-t} \sin\left(\sqrt{3}t\right).$$

We have:

$$u(0) = C_1 = 1, \ u'(0) = -C_1 + \sqrt{3}C_2 = 0,$$

so $C_2 = 1/\sqrt{3}$. Therefore,

$$\boxed{u(t) = e^{-t} \cos\left(\sqrt{3}t\right) + \frac{1}{\sqrt{3}} e^{-t} \sin\left(\sqrt{3}t\right)}$$

5. Continuation of the previous problem. Find:

(a) (4 points) Quasi frequency, quasi period.

**Solution:** Quasi Frequency: $\boxed{\sqrt{3}}$ Quasi Period: $\boxed{2\pi/\sqrt{3}}$

(b) (4 points) Amplitude and phase at time $t$.

**Solution:** Since

$$u(t) = \frac{2}{\sqrt{3}}e^{-t}\left[\frac{\sqrt{3}}{2}\cos\left(\sqrt{3}t\right) + \frac{1}{2}\sin\left(\sqrt{3}t\right)\right] = \frac{2}{\sqrt{3}}e^{-t}\cos\left[\sqrt{3}t - \frac{\pi}{6}\right],$$

the amplitude and phase are, respectively,

$$\boxed{\frac{2}{\sqrt{3}}e^{-t}} \quad \text{and} \quad \boxed{\sqrt{3}t - \frac{\pi}{6}}$$

(c) (2 points) First moment when the ball passes the equilibrium.

**Solution:** When

$$\sqrt{3}t - \frac{\pi}{6} = \frac{\pi}{2} \Rightarrow t = \boxed{\frac{2\pi}{3\sqrt{3}}}$$