Math 307A, Final Exam Name: ______________________________

Spring 2013

Instructions.

- **DON'T PANIC!** The questions on this exam are generally more interesting and challenging than the ones you saw on the midterms, but you have twice the time to solve them. If you get stuck, take a deep breath and move on to something else. Return if you have time at the end.
- There are 6 questions on 7 pages. Make sure your exam is complete.
- You are allowed **two** double-sided sheets of notes in your own handwriting. You may not use someone else's note sheet.
- You may use a simple scientific calculator, but you don't need to. No fancy calculators or other electronic devices allowed. If you didn't bring a simple calculator, then just don't use a calculator.
- It's fine to leave your answers in exact form. If you use a calculator, approximate to two decimal places.
- **Show your work**, unless instructed otherwise. If you need more space, raise your hand and I'll give you some extra paper to staple onto the back of your test.
- Don't cheat. If I see that you aren't following the rules, I will report you to UW.

Question	Points	Score
1	12	
2	12	
3	16	
4	12	
5	12	
6	15	
Total:	79	

1. (a) (2 points) Write down a differential equation modeling the motion of a **critically damped** spring-mass system. No explanation necessary.

Solution: The characteristic equation needs to have only one real root to be critically damped, and the coefficients must be positive. So one possibility is $y'' + 2y' + y = 0$.

(b) (2 points) Write a homogeneous, linear, second-order differential equation with constant coefficients for which $\{e^{-2t}, e^{3t}\}$ is a fundamental set of solutions. Again, no work necessary.

Solution: The characteristic equation needs roots -2 and 3, so $y'' - y' - 6y = 0$ (or any constant multiple thereof) works.

(c) (4 points) Write $f(t)$ in terms of step functions $u_c(t)$. Simplify your answer so each u_c appears only once (as if you were going to take Laplace transform of it).

$$f(t) = \begin{cases} \sin(t) & 0 \le t < 2 \\ t^2 - 1 & 2 \le 1 < 3 \\ 0 & t \ge 3 \end{cases}$$

Solution:

$$\begin{aligned} f(t) &= (1 - u_2(t))(\sin(t)) + (u_2(t) - u_3(t))(t^2 - 1) + u_3(t)(0) \\ &= \sin(t) + u_2(t)(t^2 - 1 - \sin(t)) + u_3(t)(1 - t^2). \end{aligned}$$

(d) (4 points) Take the Laplace transform of $g(t)$, where

$$g(t) = t + u_2(t)(t - 2) + u_3(t)t^2.$$

You may use the table. Show your work.

Solution: By the formula from class,

$$\begin{aligned} \mathcal{L}\{g(t)\} &= \mathcal{L}\{t\} + e^{-2s}\mathcal{L}\{(t+2) - 2\} + e^{-3s}\mathcal{L}\{(t+3)^2\} \\ &= \frac{1}{s^2} + e^{-2s}\frac{1}{s^2} + e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right). \end{aligned}$$

2. (12 points) Consider the nonhomogeneous differential equation

$$y'' + 9y = f(t)$$

for some function $f(t) \neq 0$. For each value of $f(t)$ below, tell me what form the **particular solution** $Y(t)$ of the differential equation would take. (For example, you might write "$A\cos(t) + Bt$", if you thought a solution would take that form.) You don't need to tell me what the solution actually is! You may leave the coefficients undetermined. Use capital letters A, B, C, and so on for the coefficients.

Remember that sometimes we need to multiply by extra powers of t. You don't need to show any work. Two points each.

$f(t)$	Form of particular solution $Y(t)$:
$5\sin(3t)$	$At\cos(3t) + Bt\sin(3t)$
$\cos(3t)\sin(3t)$	$A\cos^2(3t) + B\cos(3t)\sin(3t) + C\sin^2(3t)$
$t\cos(3t)$	$At^2\cos(3t) + Bt^2\sin(3t) + Ct\cos(3t) + Dt\sin(3t)$
$2\sin(3t) + \cos(3t) + t$	$At\cos(3t) + Bt\sin(3t) + Ct + D$
$t^2 + te^{-8t}$	$At^2 + Bt + C + Dte^{-t} + Ee^{-t}$
$e^t\cos(3t)$	$Ae^t\cos(3t) + Be^t\sin(3t)$

3. It's a nice summer day, your windows are open, and your house is 75°. You want to bake cupcakes, so you find a recipe and get the ingredients ready. The butter and eggs came from the refrigerator, and the dry ingredients are at room temperature (75°). When you mix everything together, the temperature of the cupcake batter is 60°.

   Oops! You were in such a hurry to make cupcakes that you forgot to pre-heat the oven. But you don't want to wait any longer, so you turn it on when you put the 60° cupcakes in the oven. The temperature is initially 75° in the oven, just like it is in your kitchen, and then it increases by 25° per minute until it reaches 350°. Then it just stays at 350°.

   Assume that the rate of change of the temperature $u(t)$ of the cupcakes in the oven is **equal in magnitude** to the difference between $u(t)$ and the oven temperature. (In other words: use Newton's law of cooling/warming, but with proportionality constant $k = \pm 1$.)

   (a) (12 points) Find a formula for the temperature $u(t)$ of the cupcakes in the oven at any time t. Hint: use the Laplace transform.

Solution: By Newton's law of cooling, we have $u'(t) = -(u - T)$, where $T(t)$ is the temperature in the oven. Then we have the DE

$$u' + u = \begin{cases} 75 + 25t & 0 \le t < 11 \\ 350 & t \ge 11 \end{cases}$$

In terms of unit step functions $u_c(t)$, we have

$$u' + u = 75 + 25t + u_{11}(t)(275 - 25t), u(0) = 60.$$

Take Laplace transform of both sides, letting $U = \mathcal{L}\{u\}$:

$$sU - 60 + U = \frac{75}{s} + \frac{25}{s^2} + e^{-11s}\frac{-25}{s^2}.$$

Rearrange:

$$U = \frac{75s + 25 + 60s^2}{s^2(s+1)} + e^{-11s}\frac{-25}{s^2(s+1)}.$$

Expand with partial fractions:

$$U = \left(\frac{25}{s^2} + \frac{50}{s} + \frac{10}{s+1}\right) + e^{-11s}\left(\frac{-25}{s^2} + \frac{25}{s} + \frac{-25}{s+1}\right)$$

Now take inverse Laplace, and obtain the final answer:

$$u(t) = 25t + 50 + 10e^{-t} + u_{11}(t)\left(-25(t-11) + 25 - 25e^{-(t-11)}\right)$$

(b) (2 points) Cupcakes are done cooking when they are 200°. Will your cupcakes be done before your oven reaches its final temperature of 350°? Explain.

Solution: We can just check to see what temperature the cupcakes are at $t = 11$, since that's when the oven reaches 350°. Plugging in $t = 11$, we get

$$\begin{aligned} u(11) &= 25(11) + 50 + 10e^{-11} + u_{11}(11)(0) \\ &= 275 + 50 + 10e^{-11} \\ &\simeq 325. \end{aligned}$$

Even if you didn't know what $10e^{-11}$ is, you know it's pretty small, and anyway the sum is definitely bigger than 200. So yes, the cupcakes are done long before the oven comes to temperature. Should've preheated the oven!

(c) (2 points) Write down the equation you would use to figure out when the cupcakes are 200°. Do not actually solve for t.

Solution:

$$200 = 25t + 50 + 10e^{-t}$$

The rest of this page is extra work space. Clearly label which problem(s) you're working on here, if you want me to grade it.

4. This problem is about the autonomous differential equation

$$\frac{dy}{dt} = y(y-1)^2.$$

(a) (6 points) Draw a (t, y) plane and sketch at least ten solutions to the DE. Make sure your solutions start at $t = 0$, and draw them for long enough so that their eventual behavior is clear to me. Include as many behaviors as possible.

Solution:

(b) (6 points) Find **all** solutions to the differential equation, using separation of variables. Include all equilibrium solutions. You may leave your answers in implicit form: no need to solve for y!

Solution: First, divide through by $y(y-1)^2$. Then expand using partial fractions:

$$\frac{1}{y(y-1)^2} = \frac{A}{y} + \frac{B}{y-1} + \frac{C}{(y-1)^2}.$$

The cover-up method gives $A = 1, C = 1$. To find B, just plug in $y = 2$ to get the equation $\frac{1}{2} = \frac{1}{2} + B + 1$, so $B = -1$. Finally, integrating both sides gives the implicit form

$$\ln|y| - \ln|y-1| - \frac{1}{y-1} = t + C.$$

Combining this formula with the equilibrium solutions $y = 0, y = 1$ gives all solutions.

5. (12 points) A mass of 1 kg is attached to a spring with spring constant 5. The spring exerts a damping force of 4 N when the mass is moving at 2 m/s in the opposite direction. A motor applies a force of $3\cos(3t)$ to the mass. Find a formula for the position of the mass at time t. The mass starts from rest at equilibrium. (In other words, its initial position and velocity are both zero.)

Solution: Let $y(t)$ be the position of the mass at time t. The roots of the characteristic equation are $-1 \pm 2i$, so the general homogeneous solution is

$$y_c = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t).$$

The particular solution $Y(t)$ has the form $A\cos(3t) + B\sin(3t)$. Differentiating this formula, we get

$$\begin{aligned} Y' &= -3A\sin(3t) + 3B\cos(3t) \\ Y'' &= -9A\cos(3t) - 9B\sin(3t). \end{aligned}$$

Plugging these into the DE and collecting coefficients of $\sin(3t)$ and $\cos(3t)$, we get equations

$$\begin{aligned} -4A + 6B &= 3 \\ -4B - 6A &= 0. \end{aligned}$$

So $A = -\frac{3}{13}$, $B = \frac{9}{26}$. So our formula is

$$y(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) - \frac{3}{13}\cos(3t) + \frac{9}{26}\sin(3t).$$

Now plug in the initial conditions to solve for c_1, c_2. We have $0 = y(0) = c_1 - \frac{3}{13}$. Differentiating $y(t)$ and plugging in $t = 0$, we get $0 = y'(0) = -c_1 + 2c_2 + \frac{27}{26}$. Finally, we have the solution

$$y(t) = \frac{3}{13}e^{-t}\cos(2t) - \frac{21}{52}e^{-t}\sin(2t) - \frac{3}{13}\cos(3t) + \frac{9}{26}\sin(3t).$$

6. (15 points) Solve the IVP:

$$y'' + y' + y = \begin{cases} 1 & 0 \le t < 1 \\ 0 & t \ge 1 \end{cases}$$

where $y(0) = 1, y'(0) = 1$.

Solution: In terms of step functions, the right side is $1 - u_1(t)$. Take Laplace transform of both sides:

$$\begin{aligned} s^2Y - sy(0) - y'(0) + sY - y(0) + Y &= \frac{1}{s} - e^{-s}\frac{1}{s} \\ Y &= \frac{s^2+2s+1}{s(s^2+s+1)} - e^{-s}\frac{1}{s(s^2+s+1)}. \end{aligned}$$

We'll use partial fractions to expand each of the ratios of polynomials. Also, completing the square in the denominator will be useful: $s^2 + s + 1 = (s + \frac{1}{2})^2 + \frac{3}{4}$. Write

$$\frac{s^2+2s+1}{s(s^2+s+1)} = \frac{A}{s} + \frac{B(s+\frac{1}{2}) + C}{(s+\frac{1}{2})^2 + \frac{3}{4}}.$$

The cover-up method gives us $A = 1$, and plugging in $s = 1$ and $s = -1$ gives us two equations

$$\frac{4}{3} = 1 + \frac{1.5B + C}{3}, \qquad 0 = -1 + \frac{-0.5B + C}{1}.$$

Solving this system, we get $B = 0, C = 1$. In the same way, set

$$\frac{1}{s(s^2+s+1)} = \frac{D}{s} + \frac{E(s+\frac{1}{2}) + F}{(s+\frac{1}{2})^2 + \frac{3}{4}}.$$

The cover-up method gives $D = 1$, and plugging in $s = 1, s = -1$ gives us

$$\frac{1}{3} = 1 + \frac{1.5E + F}{3}, \qquad -1 = 1 + \frac{-0.5E + F}{1}.$$

So we have $E = -1, F = -\frac{1}{2}$. Plug in these numbers and take inverse Laplace of the whole thing. Final answer:

$$\begin{aligned} y(t) = 1 &+ \frac{2}{\sqrt{3}} e^{-t/2} \sin\left(\frac{\sqrt{3}}{2}t\right) - \\ &u_1(t)\left(1 - e^{-(t-1)/2}\left(\cos\left(\frac{\sqrt{3}}{2}(t-1)\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}(t-1)\right)\right)\right) \end{aligned}$$