Math 307A, Midterm 2 Name: ______________________________

Spring 2013

Instructions.

- **DON'T PANIC!** If you get stuck, take a deep breath and go on to the next question. Come back to the question you left if you have time at the end.
- There are 4 questions on 5 pages. Make sure your exam is complete.
- You are allowed one double-sided sheet of notes in your own handwriting. You may not use someone else's note sheet.
- You may use a simple scientific calculator, but you don't need to. No fancy calculators or other electronic devices allowed. If you didn't bring a simple calculator, then just don't use a calculator.
- It's fine to leave your answers in exact form. If you use a calculator, approximate to two decimal places.
- **Show your work**, unless instructed otherwise. If you need more space, raise your hand and I'll give you some extra paper to staple onto the back of your test.
- Don't cheat. If I see that you aren't following the rules, I will report you to UW.

Question	Points	Score
1	16	
2	11	
3	9	
4	10	
Total:	46	

1. (a) (7 points) Find the general solution of the differential equation

$$y'' - 5y' + 2y = 2te^t.$$

Solution: The roots of the characteristic equation are $\frac{1}{2}\left(5 \pm \sqrt{17}\right)$, so the homogeneous solutions have the form

$$c_1 e^{\left(\frac{5}{2}+\frac{\sqrt{17}}{2}\right)t} + c_2 e^{\left(\frac{5}{2}-\frac{\sqrt{17}}{2}\right)t}.$$

A particular solution $Y(t)$ will have the form $Ate^t + Be^t$. Differentiating this, we get

$$Y' = Ae^t + Be^t + Ate^t,$$
$$Y'' = 2Ae^t + Be^t + Ate^t.$$

Plugging these into the DE and collecting coefficients, we get equations

$$-3A - 2B = 0, \qquad -2A = 2.$$

Solving, we get $A = -1$ and $B = \frac{3}{2}$. The general form of the solution is then

$$c_1 e^{\left(\frac{5}{2}+\frac{\sqrt{17}}{2}\right)t} + c_2 e^{\left(\frac{5}{2}-\frac{\sqrt{17}}{2}\right)t} - te^t + \frac{3}{2}e^t.$$

(b) (1 point) True or false: **every single solution** to the differential equation in part (a) has the form you found above.

True False

(c) (8 points) Find a fundamental set of solutions to the differential equation

$$xy'' + (2 + 2x)y' + 2y = 0.$$

Here is one solution to get you started: $y_1 = x^{-1} = \frac{1}{x}$. You don't need to check that y_1 is a solution.

Clearly label the fundamental set of solutions that is your answer.

Solution: This is a reduction of order problem, and it's not an Euler equation, so we actually do need to use the reduction of order technique. Set $y_2 = vy_1 = \frac{v}{x}$. Differentiate carefully:

$$y_2' = \frac{v'}{x} - \frac{v}{x^2},$$
$$y_2'' = \frac{v''}{x} - \frac{2v'}{x^2} + \frac{2v}{x^3}.$$

Now plug into the DE:

$$x\left(\frac{v''}{x} - \frac{2v'}{x^2} + \frac{2v}{x^3}\right) + (2 + 2x)\left(\frac{v'}{x} - \frac{v}{x^2}\right) + 2\left(\frac{v}{x}\right) = 0.$$

Collecting all the coefficients corresponding to v'', v', v, we get something much more simplified:

$$v'' + 2v' = 0.$$

This has solution $v' = e^{-2x}$, so we may also take $v = e^{-2x}$ (ignoring constant multiples). So we can take $y_2 = vy_1 = x^{-1}e^{-2x}$. So one possibility for a fundamental set is

$$\left\{\frac{1}{x}, \frac{1}{xe^{2x}}\right\}.$$

2. In this question, use $g = 9.8$ m/s^2 for gravitational acceleration.

   (a) (8 points) A 2-kilogram mass stretches a spring $\frac{g}{2} = 4.9$ meters to its equilibrium position. Marie lifts the mass up one meter and drops it at time $t = 0$. The spring has a damping force F_d of -1 N when the mass has a velocity of 1 m/s. Find a formula for the position u of the mass at time t.

   Solution: First, find k and γ. By Hooke's law at equilibrium, $mg = kL$ so $2g = k\frac{g}{2}$, so $k = 4$. Also, the formula for F_d says that $-1 = -(\gamma)(1)$, so $\gamma = 1$. So our DE is $2u'' + u' + 4u = 0$. The roots of the characteristic equation are $-\frac{1}{4} \pm \frac{\sqrt{31}i}{4}$, s the general solution is $u(t) = e^{-t/4}\left(c_1 \cos\left(\frac{\sqrt{31}}{4}t\right) + c_2 \sin\left(\frac{\sqrt{31}}{4}t\right)\right)$. Also, we'll want a formula for $u'(t)$:

$$u'(t) = e^{-t/4}\left(-c_2\frac{\sqrt{31}}{4}\sin\left(\frac{\sqrt{31}}{4}t\right) + c_2\frac{\sqrt{31}}{4}\cos\left(\frac{\sqrt{31}}{4}t\right)\right) - \frac{1}{4}e^{-t/4}\left(c_1\cos\left(\frac{\sqrt{31}}{4}t\right) + c_2\sin\left(\frac{\sqrt{31}}{4}t\right)\right)$$

   At $t = 0$, $u = -1$ and $u' = 0$. Plugging these into the formulas, we get

$$-1 = c_1, \qquad 0 = \frac{\sqrt{31}}{4}c_2 - \frac{1}{4}c_1, \qquad \text{so } c_2 = -\frac{1}{\sqrt{31}}.$$

   We have the formula

$$u(t) = e^{-t/4}\left(-\cos\left(\frac{\sqrt{31}}{4}t\right) - \frac{1}{\sqrt{31}}\sin\left(\frac{\sqrt{31}}{4}t\right)\right).$$

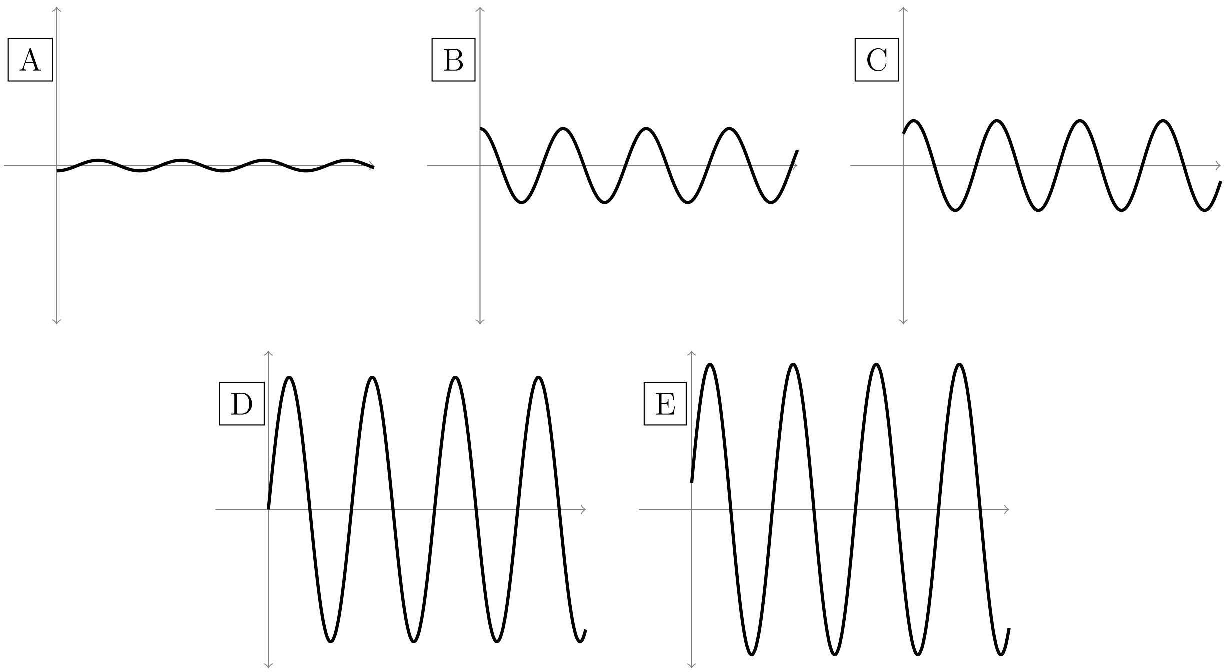
   (b) (2 points) Find a time t_1 so that for all $t \geq t_1$, the mass is within 0.1 units of equilibrium. (In other words, $|u(t)| \leq 0.1$ for every $t \geq t_1$.)

   Solution: The amplitude of this oscillation is $e^{-t/4}R$, where $R = \sqrt{1 + \frac{1}{31}} = \sqrt{\frac{32}{31}}$. This is 0.1 when $t = -4\ln\left(\sqrt{\frac{31}{32}}\left(\frac{1}{10}\right)\right) \approx 9.27$ seconds.

   (c) (1 point) What would γ have to be for this system to be critically damped? (Keep m and k the same; only change γ.)

   Solution: We'd have $\gamma^2 = 4(2)(4)$, so $\gamma = \sqrt{32}$.

3. Consider the five graphs below, labeled A–E.



(a) (5 points) Match each differential equation below to the graph **of its steady state.** Write the graph's label in the space. Each graph matches exactly one equation. You don't actually need to solve the differential equations to do this problem.

The left side is always the same in the DEs below; only the right side changes. Notice that γ is small, so there will be a large resonance effect for certain frequences.

$u'' + 0.01u' + u = \cos(0.09t)$ B

$u'' + 0.01u' + u = \cos(0.5t)$ C

$u'' + 0.01u' + u = \cos(0.99t)$ E

$u'' + 0.01u' + u = \cos(t)$ D

$u'' + 0.01u' + u = \cos(9.99t)$ A

(b) (4 points) In a few complete sentences, explain your reasoning. Be specific. Convince me that you know your answers are correct, and why.

Solution: Answers will vary, but must include most of the following: R_{max} is slightly less than 1 since γ is small; when $\omega = 1$ the graph is shifted by $\pi/2$ and is a sine function (passing through the origin); when ω is large (9.99 here) R is very small and δ is just about π; for very small ω (0.09 here), the graph is basically a cosine graph.

4. (10 points) The motion of a mass on a spring is modeled by the initial value problem

$$u'' + 2u' + 2u = 2\cos(t), \qquad u(0) = 0, u'(0) = 0.$$

Find a formula for **the steady state part** of the solution to this IVP. Put it in the standard form $R\cos(t - \delta)$.

Solution: The roots of the characteristic equation are $-1 \pm i$, so the homogeneous solution is given by $c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t)$. The particular solution will then have the form $U(t) = A\cos(t) + B\sin(t)$. Differentiating, we get

$$Y' = -A\sin(t) + B\cos(t),$$
$$Y'' = -A\cos(t) - B\sin(t).$$

Plugging in and collecting coefficients, we get equations

$$2B + A = 2, \qquad -2A + B = 0.$$

Solving these, we get $A = \frac{2}{5}$ and $B = \frac{4}{5}$. Then the steady state is

$$U(t) = \frac{2}{5}\cos(t) + \frac{4}{5}\sin(t).$$

Note that you didn't use the initial conditions for this. If I had asked for the transient part instead, you would have had to solve for c_1 and c_2.

Anyway, we can then find R and δ:

$$R = \sqrt{\left(\frac{2}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \frac{\sqrt{20}}{5} = \frac{2\sqrt{5}}{5},$$
$$\delta = \arctan\left(\frac{1}{1/2}\right) \approx 1.107.$$

No need to add π, because the coefficient of cosine is positive.

Final answer:

$$U(t) = \frac{2\sqrt{5}}{5}\cos(t - 1.107).$$