Mathematics 307 Exam

solutions

1. (a) Find the general solution of $y'' + 4y' + \frac{25}{4}y = 0$.

   Use the characteristic equation: in this case, it's $r^2 + 4r + 25/4 = 0$, and this has roots $r = -2 \pm \frac{3}{2}i$. So the general solution is

   $$\boxed{y = c_1 e^{-2t}\cos(\frac{3}{2}t) + c_2 e^{-2t}\sin(\frac{3}{2}t)}.$$

   (b) Find the general solution of $y'' + 3y' = 2$.

   First use the characteristic equation to find the solution to the associated homogeneous equation, then use undetermined coefficients to find a particular solution. The characteristic equation is $r^2 + 3r = 0$, which has roots $r = 0$ and $r = -3$. So $y_h = c_1 + c_2 e^{-3t}$. Because of the 2 on the right side of the equation, I would ordinarily try $y_p = A$. Since this is part of y_h, though, I'll try $y_p = At$ instead. Solve for A: $A = 2/3$. So the answer is

   $$\boxed{y = c_1 + c_2 e^{-3t} + \frac{2}{3}t}.$$

2. State Euler's formula.

   $$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}.$$

3. Here is a nonhomogeneous differential equation:

   $$y'' + 2y' + y = g(t).$$

   (a) What is y_h, the solution of the associated homogeneous equation?

   The characteristic equation is $r^2 + 2r + 1 = 0$, which has $r = -1$ for its only root. So

   $$\boxed{y_h = c_1 e^{-t} + c_2 t e^{-t}}.$$

   For the remaining parts, I'll tell you $g(t)$, and I want you to tell me what to try for y_p, according to the method of undetermined coefficients. You don't have to solve for the coefficients, just tell me the right form. For full credit, take your answer for part (a) into account.

   (b) If $g(t) = 7\sin 4t$, what should you try for y_p?

   This $g(t)$ has nothing to do with y_h, so I would try

   $$\boxed{y_p = A\sin 4t + B\cos 4t}.$$

(c) If $g(t) = 6e^{-t}$, what should you try for y_p?

My first guess would be Ae^{-t}, but this is in y_h. My second guess would be Ate^{-t}, but this is still in y_h. So I would try

$$\boxed{y_p = At^2e^{-t}}.$$

(d) If $g(t) = -3e^{2t} + t^3$, what should you try for y_p?

Neither summand of $g(t)$ has anything to do with y_h, so I would try

$$\boxed{y_p = Ae^{2t} + Bt^3 + Ct^2 + Dt + E},$$

(the first term because of the $3e^{2t}$, and the last 4 terms because I need to use an arbitrary degree 3 polynomial).

(e) If $g(t) = 2t^2e^t \cos 3t$, what should you try for y_p?

There is no relation between $g(t)$ and y_h. $g(t)$ is of the form (degree 2 polynomial) multiplied by $e^t \cos 3t$, so I would try

$$\boxed{y_p = (At^2 + Bt + C)e^t \cos 3t + (Dt^2 + Et + F)e^t \sin 3t}.$$

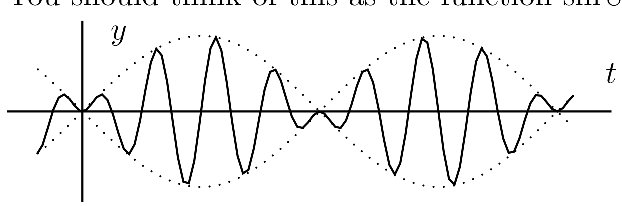
This is not the same as

$$y_{\text{bad}} = (At^2 + Bt + C)e^t(D \cos 3t + E \sin 3t),$$

and this second guess would probably not work (except possibly by coincidence). Two differences between these: the second one doesn't have as many variables, and you need all 6, not just the 5 present in the second guess. Also, if you multiply out the second one, you get something sort of like the first one, but the polynomial in front of $\cos 3t$ is not independent of the polynomial in front of $\sin 3t$. (It's as if $g(t)$ were $\sin 3t$ and you tried $A(\cos 3t + \sin 3t)$—this is no good, because the sine and cosine need different coefficients.)

4. (a) Draw a rough sketch of the function $y(t) = \sin(t)\sin(8t)$.

You should think of this as the function $\sin 8t$ oscillating between $\sin t$ and $-\sin t$:



(b) What are the differences between the graphs of the two functions

$$y_1(t) = \cos(t),$$
$$y_2(t) = 4\cos(3t - 1)?$$

There are three differences: y_2 has amplitude 4 times as large as does y_1 (so y_2 oscillates between -4 and 4, while y_1 oscillates between -1 and 1); y_2 has frequency 3 times larger than y_1's frequency (so y_2 completes one full oscillation as t goes from 0 to $2\pi/3$, while y_1 completes a full oscillation as t goes from 0 to 2π); and finally, y_2 is shifted to the right by $1/3$ (y_1 has a maximum at $t = 0$, while y_2 has its first maximum at $t = 1/3$).

5. $y_1 = t$ is one solution of the differential equation $t^2y'' + 2ty' - 2y = 0$. Find the general solution.

   Use reduction of order: let $y_2 = vy_1 = vt$, so $y_2' = v't + v$ and $y_2'' = v''t + 2v'$. Plug these into the equation; you should eventually get something like $tv'' + 4v' = 0$. This is separable and also first order linear, and you can use either method to solve it.

   Separable: $tv'' = -4v'$, so $v''/v' = -4/t$. Let $w = v'$ so $w' = v''$. Then this equation becomes $dw/w = -4dt/t$. Integrate both sides: $\ln w = -4 \ln t + c$. Solve for w: $w = e^{-4 \ln t + c} = At^{-4}$.

   First order linear: put it into standard form by making the coefficient of v'' equal to 1: $v'' + \frac{4}{t}v' = 0$. Then the integrating factor is $e^{\int (4/t)dt} = t^4$. Multiply by this and integrate: you'll get $t^4v' = c$. (Remember, the integral of 0 is c.)

   So $v' = ct^{-4}$, so $v = d_1t^{-3} + d_2$, so $y_2 = vt = d_1t^{-2} + d_2t$. This is also the general solution:

$$\boxed{y = c_1t + c_2t^{-2}}.$$

6. (a) Here is the equation of an undamped mass-spring system with an external force acting on it: $3u'' + 6u = 5 \cos \omega t$. For what value(s) of ω will the system exhibit resonance?

   You can only get resonance when the system is undamped, which this one is. In this case, you get resonance when the driving frequency ω is equal to the natural frequency $\omega_0 = \sqrt{k/m}$. In this case $m = 3$ and $k = 6$, so $\omega_0 = \sqrt{2}$, and you get resonance when $\boxed{\omega = \sqrt{2}}$.

   (b) Here is the equation of a damped mass-spring system with no external force acting on it: $2u'' + \gamma u + 3u = 0$. For what value(s) of γ will the mass exhibit oscillations?

   Oscillations mean that the solutions have sines and cosines in them, not just exponential functions. This means that the roots of the characteristic equation have to be complex. The characteristic equation is $2r^2 + \gamma r + 3 = 0$, which has roots

$$\frac{-\gamma \pm \sqrt{\gamma^2 - 24}}{4}.$$

   This will be complex if the number under the square root is negative, which happens when $\gamma^2 < 24$. Taking square roots gives the answer

$$\boxed{-\sqrt{24} < \gamma < \sqrt{24}}.$$

In a mass-spring system, the damping constant γ is never negative, so you could also say

$$\boxed{0 \le \gamma < \sqrt{24}}.$$

7. (**Bonus**) Find the general solution of the differential equation $y''' - y = t^3$.

   Use the same methods as for a second order linear nonhomogeneous equation with constant coefficients. First solve the associated homogeneous equation. The characteristic equation is $r^3 - 1 = 0$, so the roots are all numbers r so that $r^3 = 1$. There are three of these, and you can use complex numbers methods to find them: $r_1 = 1$, $r_2 = e^{2\pi i/3} = -1/2 + i\sqrt{3}/2$, $r_3 = e^{4\pi i/3} = -1/2 - i\sqrt{3}/2$. So the general solution of the associated homogeneous equation is

$$y_h = c_1 e^t + c_2 e^{-t/2}\cos(\sqrt{3}t/2) + c_3 e^{-t/2}\sin(\sqrt{3}t/2).$$

   Now find y_p. I'll try a degree three polynomial: $y_p = At^3 + Bt^2 + Ct + D$. Then $y_p''' = 6A$, so when I plug it in, I get

$$6A - (At^3 + Bt^2 + Ct + D) = t^3.$$

   So $6A - D = 0$ (constant term), $-C = 0$ (coefficient of t), $-B = 0$ (coefficient of t^2), and $-A = 1$ (coefficient of t^3). Thus $D = -6$, and $y_p = -t^3 - 6$. So the general solution is

$$\boxed{y = c_1 e^t + c_2 e^{-t/2}\cos(\sqrt{3}t/2) + c_3 e^{-t/2}\sin(\sqrt{3}t/2) - t^3 - 6}.$$