Math 307, Midterm 2 Name: ______________________________

Winter 2013

Instructions.

- **DON'T PANIC!** If you get stuck, take a deep breath and go on to the next question. Come back to the question you left if you have time at the end.
- There are 5 questions on 6 pages. Make sure your exam is complete.
- You are allowed one double-sided sheet of notes in your own handwriting. You may not use someone else's note sheet.
- You may use a simple scientific calculator, but you don't need to. No fancy calculators or other electronic devices allowed. If you didn't bring a simple calculator, then just don't use a calculator.
- It's fine to leave your answers in exact form. But you should simplify the trig functions whose values you know from the unit circle (e.g., write "0" instead of "sin(0)"). You also should simplify expressions like $e^{-2\ln t}$.
- **Show your work**, unless instructed otherwise. If you need more space, use the back of the last page. If you still need more space, just raise your hand and I'll give you more paper to staple on the back of the test.
- Don't cheat. If you do, you will receive a zero on the exam and I will report you to the university.

Question	Points	Score
1	10	
2	10	
3	8	
4	14	
5	8	
Total:	50	

1. Find the general solution to each differential equation.

(a) (4 points) $y'' + 2y = 5e^{-t}$

Solution: The solution to the homogeneous equation is

$$y_c = c_1 \cos(\sqrt{2}t) + c_2 \sin(\sqrt{2}t).$$

For a particular solution, guess $Y = Ae^{-t}$. Then $Y'' = Ae^{-t} = Y$, so we have the equation

$$3A = 5.$$

Thus the general solution is

$$y = c_1 \cos(\sqrt{2}t) + c_2 \sin(\sqrt{2}t) + \frac{5}{3}e^{-t}.$$

(b) (6 points) $y'' + 9y = 12\sin(3t)$

Solution: The roots of the characteristic equation are $\pm 3i$, so the solution to the homogeneous equation is

$$y_c = c_1 \cos(3t) + c_2 \sin(3t).$$

For the particular solution $Y(t)$, guess $Y = At\cos(3t) + Bt\sin(3t)$. The extra t is there because $A\cos(3t) + B\sin(3t)$ is a solution to the homogeneous equation. Then

$$\begin{aligned} Y' &= A\cos(3t) - 3At\sin(3t) + B\sin(3t) + 3Bt\cos(3t) \\ Y'' &= -6A\sin(3t) - 9At\cos(3t) + 6B\cos(3t) - 9Bt\sin(3t). \end{aligned}$$

Plugging these in, and collecting like terms, we get equations

$$-6A = 12, 6B = 0$$

So $A = -2, B = 0$. General solution:

$$y = c_1 \cos(3t) + c_2 \sin(3t) - 2t\cos(3t).$$

2. (10 points) Consider the initial value problem

$$x^2y'' - 2y = 0, \qquad y(1) = 5, \qquad y'(1) = 1$$

for $x > 0$. I have already found one solution to the differential equation: $y = x^2$. (You don't need to check that this is a solution.) Solve the initial value problem.

Solution: First note that the given solution doesn't work with the initial values, so we need to find another. (We know one satisfying the initial values exists, by the theorem from 3.2.) The equation is linear and homogeneous, and we're given one solution, so we may use reduction of order to find a fundamental set of solutions (and hence the general form of the solution). You also can use the method for Euler equations, since this is an Euler equation. You'll get the same answer either way. Let $y_1 = x^2$, and set $y_2 = vy_1$ for some function $v(t)$. Differentiating, we find

$$y_2' = v'x^2 + 2xv', \qquad y'' = v''x^2 + 4xv' + 2v$$

Plug these formulas into the differential equation. After a few cancellations, we have

$$v''x^4 + 4x^3v'0, \text{or} \qquad v'' + \frac{4}{x}v' = 0.$$

This is a first order linear differential equation for the function v'. The integrating factor is $\exp\left(\int(4/x)dx\right) = x^4$. Multiplying through by the integrating factor and rearranging, we have

$$\frac{d}{dt}\left(v'x^4\right) = 0, \qquad v'x^4 = c, \qquad v' = \frac{c}{x^4}.$$

Thus $v = c/x^3$ (we can absorb the $-1/3$ into the constant c). Finally, we have $y_2 = vy_1 = c/x$. Let's choose $c = 1$, for simplicity. Then we have a fundamental set of solutions

$$\left\{x^2, \frac{1}{x}\right\}$$

We know this is a fundamental set because the functions are not scalar multiples of each other. Or we can use the Wronskian:

$$W(x) = x^2\left(\frac{-1}{x^2}\right) - 2x\left(\frac{1}{x}\right) = -3 \neq 0.$$

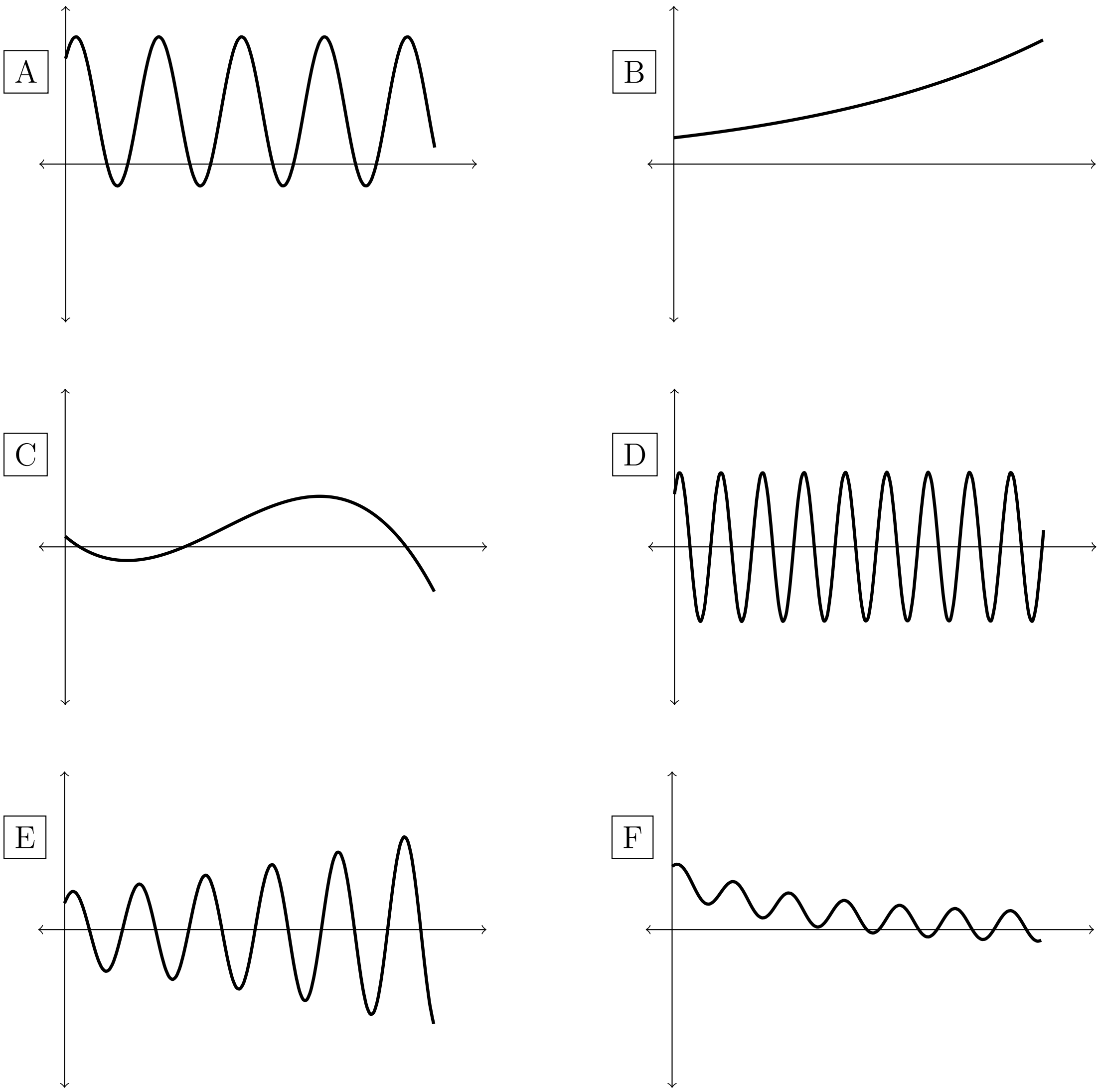
So the general form of the solution to this equation is

$$y = c_1x^2 + c_2\frac{1}{x}.$$

Now we just need to choose c_1, c_2 to satisfy the initial conditions. Using that $y(1) = 5$, we have $c_1 + c_2 = 5$. Differentiating the general form and using that $y'(1) = 1$, we have $2c_1 - c_2 = 1$. Solve to get $c_1 = 2$, $c_2 = 3$. The solution is

$$y = 2x^2 + \frac{3}{x}.$$

3. (8 points) Below are six graphs of functions $y(t)$. The t and y axes are shown. The graphs are labeled A, B, C, D, E, F.



For each differential equation below, figure out what its solutions would look like. If one of the graphs above could be a solution, write its label in the blank space. If none of the graphs could possibly be a solution, write "NONE" (don't just leave it blank). Some graphs might not be used, and some might be repeated. No work necessary.

$y'' - 4y' + 4y = 0$	B	$t^2y'' + 17t^2y = 0$	D
$y'' + 4y = 0$	D	$y'' + 4y' + 3y = 0$	NONE
$y'' + 4y' + 5y = 0$	NONE	$y'' - y = 0$	B
$2y'' - 6y' + 4y = 0$	B	$y'' - 2y' + 2y = 0$	E

4. A spring has spring constant $k = 1$. An 9-kg mass is attached to the end of the spring. Let γ be the damping constant of the system.

   (a) (8 points) Suppose $\gamma = 10$. Zach pulls the mass down from equilibrium by $\frac{1}{9}$ m, and releases it at $t = 0$ with an initial **upward** velocity of 1 m/s. Write and solve an initial value problem that models the motion of the mass. There is no external force. (Be careful with your signs!)

Solution: The problem is

$$9u'' + 10u' + u = 0, \qquad u(0) = \frac{1}{9}, \qquad u'(0) = -1.$$

The roots of the characteristic equation are $r = -\frac{1}{9}, -1$. Thus the general form of the solution is

$$y = c_1 e^{-t/9} + c_2 e^{-t}.$$

Plugging the initial conditions into y and y' gives the equations

$$c_1 + c_2 = \frac{1}{9}, \qquad -\frac{1}{9}c_1 - c_2 = -1.$$

This gives $c_1 = -1$, $c_2 = \frac{10}{9} \simeq 1.11$. So the motion of the mass is described by the equation

$$-e^{-t/9} + \frac{10}{9}e^{-t}.$$

(Problem 4 continued...)

(b) (2 points) How many times does the mass pass through its equilibrium after $t = 0$? If it does so infinitely many times, say so. Show your work.

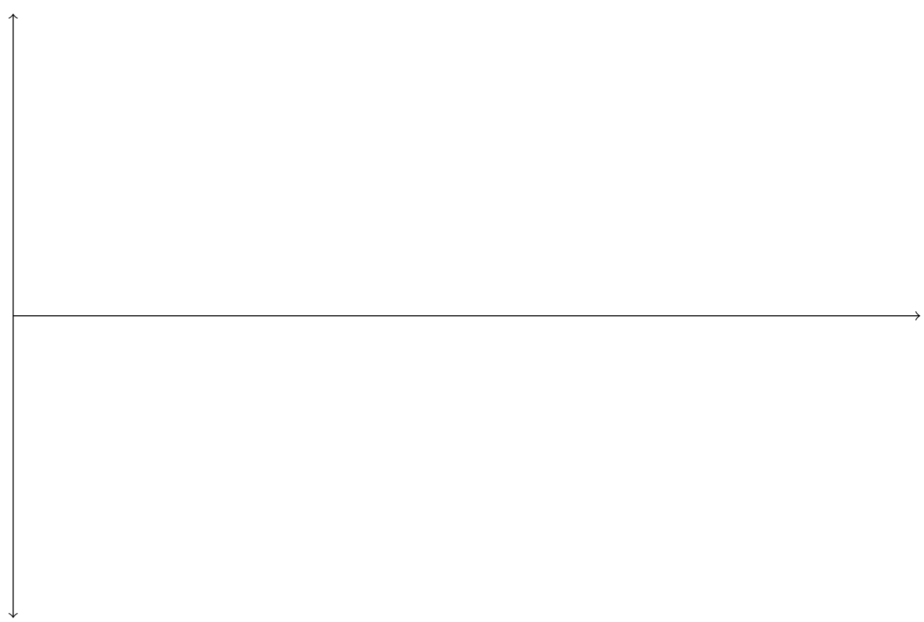
> **Solution:** It passes through equilibrium once. To see this, just set $y = 0$, and solve to get $t = \frac{9}{8}\ln\left(\frac{10}{9}\right) \approx 0.11$.

(c) (2 points) Now suppose the system is critically damped. Then what is γ?

> **Solution:** Critical damping occurs when $\gamma^2 = 4km$. So in this case, $\gamma = \sqrt{4 \cdot 1 \cdot 9} = 6$.

(d) (2 points) In this part, you don't need to compute anything. Just draw a sketch, and tell me why you think your answer is right.

Suppose γ is equal to the number you found in part (c). If Zach pulls the mass down and releases it from rest, what happens? Sketch a graph of the motion $u(t)$, starting at $t = 0$.



5. I have four identical spring-mass systems. Each system has a mass of 2 kg, a spring constant of 18 N/m, and a damping constant of $\frac{1}{9}$ N·s/m (or kg/s).

   I also have four motors, which I attach to the four spring-mass systems. The motors apply to each system an external force of $2\cos(\omega t)$, where ω is given in the table below:

System:	System 1	System 2	System 3	System 4
ω:	$\omega = 2.95$	$\omega = 4.1$	$\omega = 8.85$	$\omega = 17.9$

   The masses begin to oscillate when I turn on the motors. After 20 seconds (long enough so that the motion is basically indistinguishable from the steady state response), I look at the systems.

   (a) (3 points) Which spring-mass system is moving with the greatest amplitude? Why? You don't need to show me calculations, but you must explain your answer clearly. No credit for a guess.

   Solution: System 1. The damping constant γ is small compared to the other numbers involved, so we expect a large resonance effect for certain frequencies. The natural frequency of each spring-mass system is 3, and we know that the resonant frequency is slightly less than the natural frequency. So we choose the system with $\omega = 2.95$. The others will be much, much smaller in amplitude.

   (b) (2 points) For the system you found in part (a), tell me about the amplitude of its motion. How do you expect it to compare to 2, the amplitude of the forcing function? Why? No calculations necessary, but you must explain.

   Solution: We would expect it to be larger than 2, because the damping constant is small so there should be a resonance effect.

   (c) (3 points) Which system has the smallest amplitude? Again, explain your answer.

   Solution: System 4. Once ω is larger than the resonant frequency $\omega_{\max}$, making ω larger just makes R get increasingly small. Systems 2-4 all have frequencies larger than $\omega_{\max}$. So the smallest R corresponds to the largest ω on this list.