1. (10 total points)

(a) (5 points) Find the general solution to the following second-order differential equation:

$$3y'' + 2y' - y = 4e^{-t}\cos(t) + 2e^{-t}.$$

The characteristic equation of the homogeneous part of the DE is $3r^2 + 2r - 1 = 0$; this has solutions $r_1 = -1$ and $r_2 = \frac{1}{3}$; hence the general solution to the homogeneous part of the equation is

$$y = c_1 e^{-t} + c_2 e^{\frac{1}{3}t}.$$

To find the particular solution, we break up the forcing function into its two constituent terms and find the particular solutions for each individually. Specifically, to find the particular solution to $3y'' + 2y' - y = 4e^{-t}\cos(t)$ we guess

$$Y_1(t) = e^{-t}\left(A\cos(t) + B\sin(t)\right),$$

since we'll also need sine terms to balance coefficients. Then we have

$$Y_1' = e^{-t}\left((-A+B)\cos(t) + (-A-B)\sin(t)\right) \quad \text{and} \quad Y_1'' = e^{-t}\left((-2B)\cos(t) + (2A)\sin(t)\right),$$

so

$$\begin{aligned}
4e^{-t}\cos(t) &= 3Y_1'' + 2Y_1' - Y_1 \\
&= e^{-t}\left((-6B - 2A + 2B - A)\cos(t) + (6A - 2A - 2B - B)\sin(t)\right) \\
&= e^{-t}\left((-3A - 4B)\cos(t) + (4A - 3B)\sin(t)\right).
\end{aligned}$$

Thus we have the system of equations $-3A - 4B = 4$ and $4A - 3B = 0$, which has the solution $A = -\frac{12}{25}, B = -\frac{16}{25}$. The particular solution to this part is therefore

$$Y_1(t) = e^{-t}\left(-\frac{12}{25}\cos(t) - \frac{16}{25}\sin(t)\right).$$

Now let $Y_2(t)$ be the particular solution to the 2nd part of the nonhomogeneous equation, namely $3y'' + 2y' - y = 2e^{-t}$. Since e^{-t} is already a solution to the homogeneous DE, we guess $Y_2(t) = Ate^{-t}$. Then $Y_2' = (-At + A)e^{-t}$ and $Y_2'' = (At - 2A)e^{-t}$, so

$$\begin{aligned}
2e^{-t} = 3Y_2'' + 2Y_2' - Y_2 &= 3(At - 2A)e^{-t} + 2(-At + A)e^{-t} - Ate^{-t} \\
&= -4Ae^{-t}.
\end{aligned}$$

Equating coefficients therefore has $2 = -4A$, so $A = -\frac{1}{2}$. The particular solution to this part is thus $Y_2 = -\frac{1}{2}te^{-t}$. Finally, we combine the two particular solutions and the homogeneous solution to get the full general solution to the nonhomogeneous differential equation:

$$y = c_1 e^{-t} + c_2 e^{\frac{1}{3}t} + e^{-t}\left(-\frac{12}{25}\cos(t) - \frac{16}{25}\sin(t)\right) - \frac{1}{2}te^{-t}.$$

(b) (5 points) Find the solution $y = \phi(t)$ to the following initial value problem.

$$4y'' + y = 2\cos(t) \qquad y(0) = 0,\ y'(0) = 1.$$

The characteristic equation of the homogeneous part of the DE is $4r^2 + 1 = 0$; this has solutions $r = \pm\frac{1}{2}i$; hence the general solution to the homogeneous part of the equation is

$$y = c_1 \cos\left(\frac{1}{2}t\right) + c_2 \sin\left(\frac{1}{2}t\right).$$

For the particular solution to the full nonhomogenous DE, we would ordinarily guess $Y(t) = A\cos(t) + B\sin(t)$; however, since the homogeneous part of the DE has no y' term we don't need the sine part in our guess in order to balance coefficients. Thus we guess $Y = A\cos(t)$. We then have $Y'' = -A\cos(t)$, so

$$2\cos(t) = 4Y'' + Y = -4A\cos(t) + A\cos(t) = -3A\cos(t),$$

so we must have $2 = -3A$, i.e. $A = -\frac{2}{3}$. The particular solution is then

$$Y = -\frac{2}{3}\cos(t),$$

and so the full general solution to the differential equation is

$$y = c_1 \cos\left(\frac{1}{2}t\right) + c_2 \sin\left(\frac{1}{2}t\right) - \frac{2}{3}\cos(t).$$

2. (10 total points) Consider the initial value problem

$$(\alpha-2)y''+(3\alpha)y'+(2\alpha+1)y=0, \qquad y(0)=1, y'(0)=0$$

for a given constant α.

(a) (5 points) Find the values of α for which the solution to the IVP exhibits oscillatory behavior. For which values will the solution's oscillations be damped, constant in amplitude or exponentially growing?

The characteristic equation corresponding to this differential equation is

$$(\alpha-2)r^2+(3\alpha)r+(2\alpha+1)=0.$$

The solution will exhibit oscillatory behavior if the CE has complex roots, which in turn happens when the discriminant ('b^2-4ac') is negative. Thus to get an oscillating solution to the DE we require

$$(3\alpha)^2-4(\alpha-2)(2\alpha+1)<0,$$

or, after simplifying, $\alpha^2+12\alpha+8<0$. Now $\alpha^2+12\alpha+8$ is a quadratic in α with positive coefficient in front of the α^2 term, so it will be negative between its two roots. The quadratic formula yields $\alpha^2+12\alpha+8=0$ when $\alpha=-6\pm 2\sqrt{7}$. We therefore have that the DE exhibits oscillatory behavior for

$$-6-2\sqrt{7}<\alpha<-2+2\sqrt{7},$$

or $-11.292<\alpha<-0.708$.

Finally, for these values of α both the coefficients in front of the y'' and the y' term ($\alpha-2$ and 3α respectively) are negative; thus the CE has roots whose real parts ('$-\frac{b}{a}$') are negative. This translates into a general solution with sine and cosine terms multiplied by an exponentially decaying term. That is, the solution will be damped for all α for which the solution exhibits oscillatory behavior.

(b) (5 points) Let α be the value which maximizes the solution's quasi-frequency, and let $y(t)$ be the solution to the IVP for this value of α. Find a time t_0 beyond which the amplitude of y never exceeds 0.1, i.e. for which $|y(t)| \leq 0.1$ for all $t > t_0$.

The solution's quasi frequency is maximized when the characteristic equation has roots with imaginary parts of maximum magnitude. This in turn happens when the discriminant $\alpha^2 + 12\alpha + 8$ is the most negative. This of course happens at the turning point of the quadratic, i.e. $\alpha = -\frac{12}{2\cdot 1} = -6$. Hence the fastest oscillation happens for $\alpha = 6$. The DE then becomes $-8y'' - 18y' - 11y = 0$. After multiplying the whole equation by -1 we arrive at the IVP

$$8y'' + 18y' + 11y = 0, \qquad y(0) = 1, y'(0) = 0.$$

To find out when the amplitude of the solution decays to less than 0.1, we will write the solution in the form $y = Re^{-ct}\cos(\omega t - \delta)$ for constants R, c, ω and δ, as then we know that the solution is at most Re^{-ct} in magnitude The characteristic equation is $8r^2 + 18r + 11 = 0$, which has roots $r = -\frac{9}{8} \pm \frac{\sqrt{7}}{8} \cdot i$, so the solution to this DE can be written in the form

$$y = e^{-\frac{9}{8}t}\left(A\cos\left(\frac{\sqrt{7}}{8}t\right) + B\sin\left(\frac{\sqrt{7}}{8}t\right)\right)$$

Using the initial value $y(0) = 1$ gives us $A = 1$, while the second initial value $y'(0) = 0$ gives us $-\frac{9}{8}A + \frac{\sqrt{7}}{8}B = 0$, so $B = \frac{9}{\sqrt{7}}$.

Now recall that to convert the solution to the form $y = Re^{-ct}\cos(\omega t - \delta)$ we use $R = \sqrt{A^2 + B^2}$, so

$$R = \sqrt{1^2 + \left(\frac{9}{\sqrt{7}}\right)^2} = \sqrt{\frac{88}{7}} = 2\sqrt{\frac{22}{7}} = 3.5256.$$

We therefore know that at time t the solution is at most $2\sqrt{\frac{22}{7}}e^{-\frac{9}{8}t}$ in magnitude. To find a time beyond which the solution is always less than $\frac{1}{10}$ in magnitude, we solve for t in the equation

$$\frac{1}{10} = 2\sqrt{\frac{22}{7}}e^{-\frac{9}{8}t}.$$

Squaring both sides we get $\frac{1}{100} = \frac{88}{7}e^{-\frac{9}{4}t}$. Solving for t yields

$$t = \frac{4}{9}\ln\left(\frac{8800}{7}\right) = 3.1718.$$

We conclude that for $\alpha = -6$, the solution damps to magnitude less than 0.1 after $t = 3.1718$.

3. (10 total points) A certain vibrating system satisfies the differential equation

$$0.5y'' + 0.1y' + 2y = 3\cos(\omega_0 t)$$

where ω_0 is the natural frequency of the system.

(a) (5 points) Compute the amplitude of the system's steady-state solution.

There are two ways to approach solving this question. One way is to solve the equation fully and write the steady-state solution in the form $y = R\cos(\omega t - \delta)$ for constants R, ω and δ. However, we've done the full general case in class, and it's perfectly okay to just quote the formula for R in terms of the coefficients in the DE. To that effect, given the DE $my'' + \gamma y' + ky = F_0\cos(\omega t)$, we found in class that

$$R = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + \gamma^2\omega^2}}.$$

We have $m = \frac{1}{2}$, $\gamma = \frac{1}{10}$, $k = 2$ and $F_0 = 3$. Furthermore we know that for us $\omega = \omega_0 = \sqrt{\frac{k}{m}} = 2$. Hence

$$R = \frac{3}{\sqrt{(2 - \frac{1}{2}\cdot 2^2)^2 + \left(\frac{1}{10}\right)^2 \cdot 2^2}} = \frac{3}{\sqrt{0 + \frac{1}{25}}} = 15.$$

That is, the amplitude of the steady-state respond in this example is $R = 15$.

(b) (5 points) Suppose the forcing function's frequency is doubled to $2\omega_0$, but everything else remains the same. What does the amplitude of the steady-state solution now become?

Same setup as above, but now $\omega = 4$. Thusly:

$$R = \frac{3}{\sqrt{(2 - \frac{1}{2}\cdot 4^2)^2 + \left(\frac{1}{10}\right)^2 \cdot 4^2}} = \frac{3}{\sqrt{36 + \frac{4}{25}}} = \frac{15}{2\sqrt{226}} = 0.4989.$$

So the steady-state solution's amplitude is now much smaller, at $R = 0.4989$.

4. (10 total points) A series circuit contains a capacitor of 6.4×10^{-4} F and an inductor of 10 H. Resistance in the circuit is negligible, and the charge on the capacitor and the current in the circuit are both initially zero. At time $t = 0$ an external voltage is applied to the circuit of $125\cos(15t)$ volts.

   (a) (6 points) Formulate and solve an initial value problem using the above data to determine the charge on the capacitor at time t.

   We use the series circuit differential equation that we developed in class, i.e.

$$LQ'' + RQ' + \frac{1}{C}Q = E(t),$$

   where for us $L = 10$, $R = 0$, $1/C = 1/(6.4 \times 10^{-4}) = \frac{3125}{2}$ and $E(t) = 125\cos(15t)$. Hence we have the initial value problem

$$10Q'' + \frac{3125}{2}Q = 125\cos(15t), \qquad Q(0) = 0, Q'(0) = 0.$$

   This is precisely the case where we get beats. Checking our notes from class we see that for the initial value problem $my'' + ky = F_0\cos(\omega t)$, $y(0) = y'(0) = 0$, we can write the solution as

$$y = \left[\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{1}{2}(\omega_0 - \omega)t\right)\right] \sin\left(\frac{1}{2}(\omega_0 + \omega)t\right),$$

   where $\omega_0 = \sqrt{\frac{k}{m}}$ is the system's natural frequency. For us $\omega_0 = \sqrt{\frac{3125/2}{10}} = \frac{25}{2} = 12.5$. Furthermore for us $F_0 = 125$, $m = 10$, $\omega_0^2 - \omega^2 = \left(\frac{25}{2}\right)^2 - 15^2 = -\frac{275}{4}$, $\frac{1}{2}(\omega_0 - \omega) = -\frac{5}{4}$ and $\frac{1}{2}(\omega_0 + \omega) = \frac{55}{4}$. Hence

$$Q = \left[\frac{2 \cdot 125}{10 \cdot -\frac{275}{4}} \sin\left(-\frac{5}{4}t\right)\right] \sin\left(\frac{55}{4}t\right) = \frac{4}{11} \sin\left(\frac{5}{4}t\right) \sin\left(\frac{55}{4}t\right).$$

   If you prefer decimals, the solution can be written as

$$Q = 0.3636\sin(1.25t)\sin(13.75t).$$

   (b) (4 points) The capacitor is rated to sustain a maximum charge of 0.5 Coulombs. Is this circuit safe given the above setup, or will it burn out?

   This answer is straightforward given the way that we've written the solution above. Since both the sine terms never exceed 1 in magnitude, we see that the solution never exceeds $\frac{4}{11}$ in magnitude at any given point in time. Since this is less than $\frac{1}{2}$, we conclude that the circuit is safe, and will not burn out.

5. (10 points) (10 total points) An object of unknown mass is placed on a flat surface and attached to a horizontal spring with spring constant 2.5 kg/s^2. The damping constant in the system is precisely 1 kg/s. The object is stretched 1 meter to the right of its equilibrium position and released with zero initial velocity. The damped oscillations of its subsequent motion are observed to have a quasi-period of $\frac{20}{7}\pi$ seconds.

   What is the mass of the object?

   Here we have the initial value problem

$$my'' + y' + \frac{5}{2}y = 0, \qquad y(0) = 1,\ y'(0) = 0.$$

   The corresponding characteristic equation is

$$mr^2 + r + \frac{5}{2} = 0,$$

   with roots

$$r = \frac{-1 \pm \sqrt{1^2 - 4 \cdot m \cdot 5/2}}{2m} = \frac{-1}{2m} \pm \frac{1}{2m}\sqrt{1 - 10m}.$$

   We are told the solution exhibits oscillatory behavior, so the thing under the square root sign must be negative. Hence the roots to the CE can be written as

$$\frac{-1}{2m} \pm \frac{\sqrt{10m-1}}{2m} \cdot i.$$

   This means the solution will contain sine and cosine terms with radial quasi-frequency ω, where $\omega = \frac{\sqrt{10m-1}}{2m}$.

   On the other hand, if T is the quasi-period, then $\omega = \frac{2\pi}{T}$; hence

$$\omega = \frac{2\pi}{\frac{20}{7}\pi} = \frac{7}{10}.$$

   Thus we must have that $\frac{\sqrt{10m-1}}{2m} = \frac{7}{10}$. It now remains to solve for m. Cross-multiplying to clear denominators we get

$$5\sqrt{10m-1} = 7m,$$

   so after squaring both sides we have $25(10m-1) = 49m^2$, or

$$49m^2 - 250m + 25 = 0.$$

   This quadratic has the solutions $m = 5$ or $m = \frac{5}{49}$.

   Going back to our original differential equation, we see that 5 and $\frac{5}{49}$ are both valid (non-negative) values for the object's mass such that the quasi-frequency of the solution's oscillations is $\frac{7}{10}$. We therefore conclude that either $m = 5$ kg or $m = \frac{5}{49}$ kg, and that there is no way beyond this to tell given the problem setup.