1. (10 points) Find the explicit solution to the initial value problem

$$y' + t^2 y - t^2 = 0, \qquad y(0) = -1$$

Your answer should be in the form $y = g(t)$, where $g(t)$ contains no undetermined constants.

This first order equation is both linear and separable; we will solve it using integrating factors. In standard linear form the equation is $y' + t^2 y = t^2$. The integrating factor is therefore

$$\mu(t) = e^{\int t^2\,dt} = e^{\frac{1}{3}t^3}.$$

The general solution to the equation is then

$$\begin{aligned} y &= \frac{1}{\mu(t)}\left(\int \mu(t) g(t)\,dt + C\right) \\ &= e^{-\frac{1}{3}t^3}\left(C + \int t^2 e^{\frac{1}{3}t^3}\,dt\right) \\ &= e^{-\frac{1}{3}t^3}\left(C + e^{\frac{1}{3}t^3}\right) \\ &= Ce^{-\frac{1}{3}t^3} + 1. \end{aligned}$$

Hence the general solution is

$$y = 1 + Ce^{-\frac{1}{3}t^3}.$$

2. (10 points) Use the method of undetermined coefficients to find the **particular solution** to the following differential equation:

$$y'' + 9y = te^{-t} - 1.$$

Your answer should be a function $Y(t)$ with no undetermined constants in it.

Particular solutions are additive, so we know the particular solution will be the sum of the particular solutions to $y'' + 9y = -1$ and $y'' + 9y = te^{-t}$. For the first equation we have a constant forcing function, and since constants don't obey the homogeneous part of the DE we therefore guess a constant for this part of the particular solution. That is, guess

$$Y_1(t) = A$$

for some value of $A$. Then $Y_1' = Y_1'' = 0$, so when we plug $Y_1$ back into the equation we get

$$0 + 9 \cdot 0 + 9A = -1.$$

Solving for $A$ gives us $A = -\frac{1}{9}$, so $Y_1 = -\frac{1}{9}$.

The second equation $y'' + 9y = te^{-t}$ has a forcing function that is a linear polynomial times an exponential. Since this doesn't obey the homogeneous part of the DE, we guess a general linear polynomial times an exponential of the same exponent, i.e.

$$Y_2(t) = (At + B)e^{-t}.$$

Note that we will need two undetermined constants in our guess: when we take derivatives of the guess we will get [constant]$\times e^{-t}$ terms appear, and if we don't have one in our guess we won't be able to balance coefficients.

Now $Y_2' = ((-A)t + (A - B))\,e^{-t}$ and $Y_2'' = ((A)t + (-2A + B))\,e^{-t}$, so plugging $Y_2$ back into the DE gives us

$$\begin{aligned} Y_2'' + 9Y_2 &= (At + (-2A + B))\,e^{-t} + 9(At + B)e^{-t} \\ &= 10Ate^{-t} + (-2A + 10B)e^{-t}. \end{aligned}$$

Since this must equal $te^{-t} = 1 \cdot te^{-t} + 0 \cdot e^{-t}$ we therefore have that $10A = 1$ and $-2A + 10B = 0$. Solving this system of equations gives us $A = \frac{1}{10}$ and $B = \frac{1}{50}$. Hence

$$Y_2(t) = \left(\frac{1}{10}t + \frac{1}{50}\right) e^{-t}.$$

Finally, combining the two particular solutions gives us the solution $Y(t)$ to the complete nonhomogeneous differential equation. That is,

$$Y(t) = -\frac{1}{9} + \frac{1}{50}\,(5t + 1)\,e^{-t}.$$

3. (10 total points) Consider following initial value problem:

$$\frac{dy}{dx} = \cos^2(x)\cos^2(y), \qquad y(0) = y_0,$$

where $y_0$ is a given constant.

(a) (2 points) By using the existence and uniqueness theorem for first-order differential equations, find the values of $y_0$, if any, for which the IVP is *not* guaranteed a unique solution in some time interval about $x = 0$.

Recall that for a first-order DE $\frac{dy}{dx} = f(x,y)$, a unique solution is guaranteed to exist about the initial condition point if both $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous near the initial condition point. Now $f(x,y) = \cos^2(x)\cos^2(y)$ is a continuous everywhere infinitely differentiable function of both $x$ and $y$, so both $f(x,y)$ and its partial derivative $\frac{\partial f}{\partial y} = -2\cos^2(x)\sin(y)\cos(y)$ are defined and continuous for all values of $x$ and $y$. Therefore by the existence and uniqueness theorem we can conclude that we'll never run into trouble, i.e. There are no values of $y(0) = y_0$ for which we won't get a unique solution to the DE.

(b) (6 points) Solve the IVP for the initial value $y_0 = 0$. [Hint: $\frac{d}{dy}\tan(y) = \sec^2(y)$ ]

This equation is separable. Putting the $y$ stuff on the left and the $x$ stuff on the right gives us

$$\sec^2 y\, dy = \cos^2(x)\, dx$$

The integral of the left hand side is $\tan(y)$, while for the right hand side we have

$$\begin{aligned}\int \cos^2(x)\, dx &= \int \frac{1}{2}(1+\cos(2x))\, dx \\ &= \frac{1}{2}\left(x + \frac{1}{2}\sin(2x)\right) + C \\ &= \frac{1}{2}(x + \sin(x)\cos(x)) + C.\end{aligned}$$

Thus we get $\tan(y) = \frac{1}{2}(x+\sin(x)\cos(x)) + C$. Applying the initial condition $y(0) = 0$ yields $0 = 0 + C$, so $C = 0$. Therefore the solution to the IVP is

$$y = \arctan\left(\frac{1}{2}(x+\sin(x)\cos(x))\right).$$

(c) (2 points) Let $y = \phi(x)$ be the solution you found in part (b). What is the limiting value of the solution i.e. what is $\lim_{x\to\infty}\phi(x)$? Be sure to justify your answer.

Let $w = \frac{1}{2}(x+\sin(x)\cos(x))$ Observe that $w \to \infty$ as $x \to \infty$. Thus $\lim_{x\to\infty}\phi(x) = \lim_{w\to\infty}\arctan(w) = \frac{\pi}{2}$; that is, the limiting value of the solution is $\frac{\pi}{2}$.

Alternatively we can note that $y = \frac{\pi}{2}$ is an equilibrium solution for the DE, but no other equilibrium solutions exist for $0 < y < \frac{\pi}{2}$. By looking at the right hand side of the DE we can see that $\phi(t)$ is never decreasing, so it must limit to this value.

4. (10 points) In each part of this question you are given a function $y(t)$ which is the general solution to a constant-coefficient homogeneous 2nd-order differential equation. Write down the differential equation that that function satisfies. Your answer should be a DE in the form $ay'' + by' + cy = 0$ for some values of $a$, $b$ and $c$.

   Each part is worth 2 points. You don't need to show your working to get full credit for this question.

   (a) $y(t) = c_1 \cos(6t) + c_2 \sin(6t)$

   This solution corresponds to pure oscillation with no damping i.e. $my'' + ky = 0$ for some positive values of $k$ and $m$. Furthermore we see that the fundamental frequency is $\omega_0 = \sqrt{\frac{k}{m}} = 6$, so $k = 36m$. We might as well take $m = 1$ since it's not stipulated anywhere, which leaves us with the DE

$$y'' + 36y = 0.$$

   (b) $y(t) = c_1 e^{2t} + c_2 e^{-3t}$

   This corresponds to a DE whose characteristic equation has roots $r = 2$ and $r = -3$. That is, the characteristic equation is $(r-2)(r+3) = r^2 + r - 6 = 0$. By comparing coefficients we see that the differential equation that $y(t)$ satisfies is therefore

$$y'' + y' - 6y = 0.$$

   (c) $y(t) = c_1 + c_2 t$

   This is a general linear equation. Since all linear functions have double derivative equal to zero, we have that $y(t)$ satisfies the DE

$$y'' = 0.$$

   (d) $y(t) = c_1 e^{-t} + c_2 t e^{-t}$

   We recognize that this is the general solution of a DE whose characteristic has repeated roots. We note that since $e^{-t}$ is a solution the repeated root must be $r = -1$. In other words $(r+1)^2 = r^2 + 2r + 1 = 0$, so we get the DE

$$y'' + 2y' + y = 0.$$

   (e) $y(t) = c_1 e^{4t} \cos(2t) + c_2 e^{4t} \sin(2t)$

   Both parts of the general solution have exponential growth with growth rate 4 and oscillation with radial frequency 2. We recognize that this comes from a DE whose characteristic equation has roots $r = 4 \pm 2i$. Hence the CE is $(r-4+2i)(r-4-2i) = (r-4)^2 - (2i)^2 = r^2 - 8r + 20 = 0$, so we get the DE

$$y'' - 8y' + 20y = 0.$$

5. (10 points) Compute the inverse Laplace transform of the following function. Your answer should be a function $f(t)$. You may quote any formula or rule given in the Laplace transform formula sheet at the back of the exam paper.

$$F(s) = \frac{s^2+2s-2}{s^3-s}$$

We need to rewrite $F(s)$ using partial fractions so that we can get it into a form where we can take recognizable inverse Laplace transforms of all the constituent parts.

Observe that $s^2 - s = s(s-1)(s+1)$, so we can write

$$\frac{s^2+2s-2}{s^3-s} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}.$$

Clearing denominators gives us

$$s^2+2s-2 = A(s-1)(s+1) + Bs(s+1) + Cs(s-1).$$

When we evaluate the above equation at $s=0$ we get $-2=-A$, so $A=2$.
When we evaluate the above equation at $s=1$ we get $1=2B$, so $B=\frac{1}{2}$.
When we evaluate the above equation at $s=-1$ we get $-3=2C$, so $C=-\frac{3}{2}$. Hence

$$\frac{s^2+2s-2}{s^3-s} = 2\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{1}{s-1} - \frac{3}{2}\cdot\frac{1}{s+1}.$$

Finally, recall that $\mathscr{L}^{-1}\left[\frac{1}{s}\right] = 1$, while $\mathscr{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$. Thus by linearity of the inverse Laplace transform we arrive at

$$\mathscr{L}^{-1}\left[\frac{s^2+2s-2}{s^3-s}\right] = 2 + \frac{1}{2}e^{t} - \frac{3}{2}e^{-t}.$$

[Note: many of you listed the answer as

$$\mathscr{L}^{-1}\left[\frac{s^2+2s-2}{s^3-s}\right] = 2 - \cosh(t) + 2\sinh(t).$$

This is also correct, as $\cosh(t) = \frac{e^t+e^{-t}}{2}$ and $\sinh(t) = \frac{e^t-e^{-t}}{2}$.]

6. (10 total points) A series circuit contains a capacitor of $4 \times 10^{-4}$ F and an inductor of 1 H. The charge on the capacitor and the current in the circuit are both initially zero. Starting at time $t = 0$ an external voltage of $300\cos(40t)$ volts is applied to the circuit, where $t$ is measured in seconds. Resistance is negligible. Consider the differential equation governing the charge $Q(t)$ in Coulombs on the capacitor as a function of time.

   To answer the following questions you may use known formulae to save time, but if so be sure to state the formula as you've seen it in class.

   (a) (3 points) Write down an initial value problem describing the charge on the capacitor as a function of time.

   Recall the standard series circuit differential equation:

   $$LQ'' + RQ' + \frac{1}{C}Q = E(t),$$

   where all the quantities are expressed in SI units, as they are above. For us $L = 1$ H, $R = 0\ \Omega$, $\frac{1}{C} = \frac{1}{0.0004} = 2500$ F and $E(t) = 300\cos(t)$ V. Furthermore, zero initial charge on the capacitor and no initial current give us the initial conditions $Q(0) = 0$ and $Q'(0) = 0$ respectively. Putting this all together we get the IVP

   $$Q'' + 2500Q = 300\cos(40t), \qquad Q(0) = 0,\ Q'(0) = 0.$$

   (b) (2 points) What is the natural frequency of this system?

   Quoting our know formulae we have that in the system $my'' + \gamma y' + ky = g(t)$, natural frequency $\omega_0$ is given by $\omega_0 = \sqrt{\frac{k}{m}}$. For us this translates to

   $$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{2500} = 50 \text{ rad/sec.}$$

   (c) (2 points) The solution to the IVP above will exhibit beats. What is the beat (angular) frequency of the solution in radians/sec?

   Quoting our known formula, we see that beat frequency $\omega_1 = \frac{1}{2}(\omega_0 - \omega)$, where $\omega_0$ is the natural frequency of the system, and $\omega$ is the forcing function frequency. We therefore have that

   $$\omega_1 = \frac{1}{2}(50 - 40) = 5 \text{ rad/sec.}$$

   Note that you may have $\omega$ and $\omega_0$ swapped in the above formula; if you do you come out with a beat frequency as $-5$ rad/sec. This is not an issue; we usually normalize so that beat frequency is positive (we can do this by shifting the sinusoidal function by $\pi$ radians, or by sticking the minus sign on the outside), but this answer would also be correct.

(d) (3 points) What is the maximum amount of charge that the capacitor will hold?

Recall that the solution in the case of beats can be written as

$$Q = [R\sin(\omega_1 t)]\sin(\omega_2 t),$$

where $\omega_1$ is the beat frequency mentioned in the previous part of the question, and $\omega_2 = \frac{1}{2}(\omega_0 + \omega)$ is the frequency of rapid oscillation. Since both the sinusoidal terms vary between -1 and 1, the maximum charge on the capacitor must then be the oscillation amplitude $R$ in the above equation.

Consulting our list of known formulae again we see that in the case of the DE $my'' + \gamma y' + ky = F_0\cos(\omega t)$ the oscillation amplitude is given by

$$R = \frac{2F_0}{m(\omega_0^2 - \omega^2)},$$

so for us we have

$$R = \frac{2 \cdot 300}{1(50^2 - 40^2)} = \frac{600}{900} = \frac{2}{3} \text{ Coulombs.}$$

7. (10 total points) A heavy block of mass 1 kg is placed on a flat surface and attached to a horizontal spring. When the block is displaced 25 cm to the right of its equilibrium position the spring exerts a restoring force of 1 Newton to the left. Friction acts on the block proportional to its velocity such that when its speed is 1 m/s the block experiences a drag force of 4 Newtons.

   At time $t = 0$ seconds the block is at its equilibrium position traveling with a velocity of 1 m/s (i.e. traveling to the right). At $t = 1$ seconds a motor is switched on which exerts a force of $t - 1$ Newtons on the block. At $t = 4$ seconds the motor is switched off, and no external force acts on the block from thereon.

   (a) (2 points) Rewrite the forcing function $g(t)$ using Heaviside functions $u_c(t)$. Your answer should be expressible as a linear combination of $u_c(t)$'s each multiplied by some function of $t$.

   We write $g(t)$ in terms of Heaviside functions by building up a series of functions $g_i(t)$ which agree with $g(t)$ for larger and larger time intervals.

   The function $g(t)$ is identically zero until $t = 1$, so start with $g_1(t) = 0$; thus $g(t) = g_1(t)$ for $0 \leq t < 1$.
   Then $g(t)$ becomes $t - 1$ at $t = 1$, so let $g_2(t) = g_1(t) + u_1(t)(t-1) = u_1(t)(t-1)$. We then have $g(t) = g_2(t)$ for $0 \leq t < 4$.
   Finally $g(t)$ becomes 0 for $t \geq 4$, so let $g_3(t) = g_2(t) + u_4(t)[0 - (t-1)]$. Now $g(t) = g_3(t)$ for all $t \geq 0$, so we have found that (after simplifying)

$$g(t) = u_1(t)(t-1) - u_4(t)(t-1).$$

   (b) (3 points) Establish an initial value problem that models the position of the block for $t \geq 0$.

   The standard damped oscillator equation is

$$my'' + \gamma y' + ky = g(t),$$

   where $m$ is the mass of the object being modeled, $\gamma$ is the damping coefficient, $k$ is the spring constant, and $g(t)$ is the forcing function. The latter is exactly the $g(t)$ we've found in the part of the question; the mass $m$ is given as 1 kg, so it remains to find $\gamma$ and $k$.

   Now in a spring system as we have here the restoring force $F_s$ exerted by the spring is proportional to the displacement of the object relative to its equilibrium position. In other works $F_s = ky$, where $k$ is the spring constant. We have $F_s = 1$ N when $y = \frac{1}{4}$ m, so we must have that $k = 4$ kg/s$^2$.

   And for damping we note that the drag force $F_f$ is proportional to velocity, i.e. $F_f = \gamma y'$, where $\gamma$ is the damping constant. We're told that $F_f = 4$ when $y' = 1$; the damping constant is thus $\gamma = 4$ kg/s.

   Finally, because the block is initially at its equilibrium position traveling to the right at 1 m/s, we discern the initial conditions $y(0) = 0$, $y'(0) = 1$. Putting these all together we arrive at the IVP

$$y'' + 4y' + 4y = u_1(t)(t-1) - u_4(t)(t-1), \qquad y(0) = 0, \;\; y'(0) = 1.$$

(c) (5 points) Let $y = \phi(t)$ be the solution to the IVP above. Compute the Laplace transform $\Phi(s)$ of the solution as a function of $s$.

[NB: you do not need to fully solve the IVP to answer this part of the question.]

To compute $\Phi(s)$ we take Laplace transforms of both sides of the DE established in the previous part of the question. Using our list of known rules, we transform the left hand side as follows:

$$\begin{aligned}\mathscr{L}\left[\phi''\right] + 4\mathscr{L}\left[\phi'\right] + 4\mathscr{L}\left[\phi\right] &= \left(s^2\Phi - s\phi(0) - \phi'(0)\right) + 4\left(s\Phi - \phi(0)\right) + 4\Phi \\ &= (s^2+4s+4)\Phi - 0\cdot s - 1 - 4\cdot 0 \\ &= (s+2)^2\Phi - 1.\end{aligned}$$

Hitting the right hand side with Laplace yields

$$\mathscr{L}\left[u_1(t)(t-1) - u_4(t)(t-1)\right] = \mathscr{L}\left[u_1(t)(t-1)\right] - \mathscr{L}\left[u_4(t)(t-1)\right].$$

The first Laplace transform doesn't give us any trouble. We invoke that if $\mathscr{L}\left[f(t)\right] = F(s)$, then $\mathscr{L}\left[u_c(t)f(t-c)\right] = e^{-cs}F(s)$; here $c = 1$ and $f(t) = t$, so we get

$$\mathscr{L}\left[u_1(t)(t-1)\right] = e^{-s}\mathscr{L}\left[t\right] = \frac{e^{-s}}{s^2}.$$

However, to compute $\mathscr{L}\left[u_4(t)(t-1)\right]$ we must first put $t-1$ in the form of $f(t-4)$, as this is the format that we know how to deal with. But $t-1 = (t-4)+3$, so

$$\begin{aligned}\mathscr{L}\left[u_4(t)(t-1)\right] &= \mathscr{L}\left[u_4(t)\left((t-4)+3\right)\right] \\ &= e^{-4s}\mathscr{L}\left[t+3\right] \\ &= e^{-4s}\left(\frac{1}{s^2} + \frac{3}{s}\right) \\ &= \frac{e^{-4s}(1+3s)}{s^2}.\end{aligned}$$

Hence the Laplace transform of the right hand side of the DE is $\frac{e^{-s}-e^{-4s}(1+3s)}{s^2}$.

Finally, by equating the left and right hand sides of the transformed equation we get

$$(s+2)^2\Phi - 1 = \frac{e^{-s} - e^{-4s}(1+3s)}{s^2}.$$

Solving for $\Phi$ gives us

$$\Phi = \frac{e^{-s} - e^{-4s}(1+3s)}{s^2(s+2)^2} + \frac{1}{(s+2)^2}.$$

Note: we could try simplify the above equation a bit, but it's probably wasted effort seeing as we'll need to break up the right hand side using partial fractions when taking inverse Laplace transforms. So I'm happy if you just leave the answer as is.

8. (10 total points + 4 bonus points) A two-way pump attached attached to a reservoir pumps water into and then out of the reservoir at a rate of $2000\sin(t)$ liters per hour, where $t$ is measured in hours. At time $t = 0$ a valve at the bottom of the reservoir is opened and it begins to drain at a rate proportional to the amount of water in the reservoir. The reservoir initially contains 10000 liters of water, and the initial outflow rate is measured to be 1000 liters per hour.

   (a) (3 points) Establish an initial value problem that models the volume of water in the reservoir at time $t$.

   This is a flow rate problem. Let $y(t)$ be the number of liters of water in the reservoir at time $t$, where $t$ is measured in hours since the the valve is opened. Recall that the rate of change of $y$ equals rate in minus rate out. In this instance the "rate in" is the pump flow rate, i.e. $2000\sin(t)$ liters per hour, while the rate out is the rate at which water escapes through the valve at the bottom of the tank.

   We are told the outflow rate through the valve is proportional to the number of liters currently in the reservoir; that is, outflow rate $= -ky$, where $k$ is some proportionality constant. We therefore have that

$$\frac{dy}{dt} = 2000\sin(t) - ky.$$

   We are not told what $k$ is, but we are told that the initial outflow rate is 1000 liters per hour. Since the pump flow rate is $2000\sin(0) = 0$, this must mean that the initial flow rate through the valve is $-1000$ liters per hour. We therefore have that at time zero $-1000 = -k \cdot 10000$, so $k = \frac{1}{10}$.

   Putting this and the initial condition $y(0) = 10000$ together, we get the IVP

$$\frac{dy}{dt} = -\frac{1}{10}y + 2000\sin(t), \qquad y(0) = 10000.$$

   (b) (7 points) Solve the initial value problem to find the number of liters of water in the reservoir at time $t$.

   This equation is linear. In standard form it is $y' + \frac{1}{10}y = 2000\sin(t)$. We could solve this IVP using integrating factors, but since we've already used that method in a previous question, we'll use Laplace transforms here instead.

   Let $y = \phi(t)$ be the solution to this IVP, and let $\Phi(s) = \mathscr{L}[\phi(t)]$. We now take Laplace transform of both sides of the equation. On the left hand we get

$$\begin{aligned}\mathscr{L}[\phi'] + \frac{1}{10}\mathscr{L}[\phi] &= (s\Phi - \phi(0)) + \frac{1}{10}\Phi \\ &= \left(s + \frac{1}{10}\right)\Phi - 10000.\end{aligned}$$

On the right hand side we get

$$\mathscr{L}\left[2000\sin(t)\right] = 2000 \cdot \frac{1}{s^2+1}.$$

Equating the two sides and solving for $\Phi$ gives us

$$\Phi = 10000 \cdot \frac{1}{(s+\frac{1}{10})} + 2000 \cdot \frac{1}{(s+\frac{1}{10})(s^2+1)}.$$

All that remains then is to compute the inverse Laplace transform of $\Phi(s)$. Note that we can use partial fractions on $\frac{1}{(s+\frac{1}{10})(s^2+1)}$ to break it up into a sum of terms with only $s+\frac{1}{10}$ or $s^2+1$ in the denominator. That is

$$\frac{1}{(s+\frac{1}{10})(s^2+1)} = \frac{A}{s+\frac{1}{10}} + \frac{Bs+C}{s^2+1}$$

for some values of $A, B$ and $C$. Clearing denominators we get

$$1 = A(s^2+1) + (Bs+C)\left(s+\frac{1}{10}\right).$$

Evaluating at $s = -\frac{1}{10}$ yields $1 = A\cdot(\frac{1}{100}+1)$, so $A = \frac{100}{101}$.
Evaluating at $s = 0$ yields $1 = A + \frac{1}{10}C$, so $C = \frac{10}{101}$;
Evaluating at $s = 1$ yields $1 = 2A + \frac{11}{10}(B+C)$, so solving for $B$ we get $B = -\frac{100}{101}$. That is, we have

$$\frac{1}{(s+\frac{1}{10})(s^2+1)} = \frac{10}{101}\left(10\cdot\frac{1}{s+\frac{1}{10}} - 10\cdot\frac{s}{s^2+1} + \frac{1}{s^2+1}\right).$$

We therefore have that the inverse Laplace transform of $\Phi$ is

$$\begin{aligned}\mathscr{L}^{-1}\left[\Phi\right] &= 10000\,\mathscr{L}^{-1}\left[\frac{1}{s+\frac{1}{10}}\right] + \frac{20000}{101}\left(10\,\mathscr{L}^{-1}\left[\frac{1}{s+\frac{1}{10}}\right] - 10\,\mathscr{L}^{-1}\left[\frac{s}{s^2+1}\right] + \mathscr{L}^{-1}\left[\frac{1}{s^2+1}\right]\right)\\ &= 10000e^{-\frac{1}{10}t} + \frac{20000}{101}\left(10e^{-\frac{1}{10}t} - 10\cos(t) + \sin(t)\right).\end{aligned}$$

So after collecting terms we conclude that the amount of water in the reservoir at time $t$ is given by

$$y = \phi(t) = \frac{10000}{101}\left(121e^{-\frac{1}{10}t} - 20\cos(t) + 2\sin(t)\right) \text{ liters.}$$

[Question continued on the next page]

(c) (4 bonus points) Estimate the point in time when the reservoir first runs dry.

We want to find the first $t$ such that $\phi(t) = 0$. That is, solve for $t$ when

$$121e^{-\frac{1}{10}t} - 20\cos(t) + 2\sin(t) = 0.$$

Note that we may divide through by that ugly $\frac{10000}{101}$ coefficient, since it won't affect where the function hits zero.

Now this is not an equation where one can solve for $t$ explicitly, so we'll have to make some approximations. Note that the exponential decay constant is small competed to the oscillation frequency; we therefore expect that in the time take for the oscillation part to go between its minimum and maximum values, the exponential part of the function will not have decayed very much. In other words, we can think of the function as relatively rapid oscillation about some slowly decreasing equilibrium.

The result is that the first root of the equation should be near the point in time where $121e^{-\frac{1}{10}t} - R$ first dips below zero, where $R$ is the amplitude of the oscillation part. Recall that $R = \sqrt{A^2 + B^2}$, where $A$ and $B$ are the amplitudes of the cos and sin functions, so we have $R = \sqrt{20^2 + 2^2} = 2\sqrt{101}$. We thus should solve for when

$$121e^{-\frac{1}{10}t} - 2\sqrt{101} = 0.$$

Solving for $t$ in this equation gives us

$$t = 10\log\left(\frac{121}{2\sqrt{101}}\right) = 5\log\left(\frac{14641}{404}\right) \approx 17.95 \text{ hours.}$$

We conclude that the reservoir empties at just under the 18 hour mark.

In reality the first time $y(t) = 0$ happens a bit later than this - as the oscillation doesn't hit its minimum value at exactly the moment when the decay part of the equation comes within $R$ of zero. Using Mathematica to numerically find the root we get $t \approx 18.44$ hours, so about a half hour after our prediction. Our estimate is therefore within 3 percent of the actual value, which for most on-the-fly real world problems is good enough.

# Table of Laplace Transforms

In this table, $n$ always represents a positive integer, and $a$ and $c$ are real constants.

| $f(t) = \mathscr{L}^{-1}[F(s)]$ | $F(s) = \mathscr{L}(f(t))$ | |
|---|---|---|
| $1$ | $\frac{1}{s}$ | $s > 0$ |
| $e^{at}$ | $\frac{1}{s-a}$ | $s > a$ |
| $t^n$, $n$ a positive integer | $\frac{n!}{s^{n+1}}$ | $s > 0$ |
| $t^n e^{ct}$, $n$ a positive integer | $\frac{n!}{(s-c)^{n+1}}$ | $s > c$ |
| $t^a$, $a > -1$ | $\frac{\Gamma(a+1)}{s^{a+1}}$ | $s > 0$ |
| $\cos(at)$ | $\frac{s}{s^2+a^2}$ | $s > 0$ |
| $\sin(at)$ | $\frac{a}{s^2+a^2}$ | $s > 0$ |
| $\cosh(at)$ | $\frac{s}{s^2-a^2}$ | $s > \|a\|$ |
| $\sinh(at)$ | $\frac{a}{s^2-a^2}$ | $s > \|a\|$ |
| $e^{ct}\cos(at)$ | $\frac{s-c}{(s-c)^2+a^2}$ | $s > c$ |
| $e^{ct}\sin(at)$ | $\frac{a}{(s-c)^2+a^2}$ | $s > c$ |
| $u_c(t)$ | $\frac{e^{-cs}}{s}$ | $s > 0$ |
| $u_c(t)f(t-c)$ | $e^{-cs}F(s)$ | |
| $e^{ct}f(t)$ | $F(s-c)$ | |
| $f(ct)$ | $\frac{1}{c}F\left(\frac{s}{c}\right)$ | $c > 0$ |
| $f^{(n)}(t)$ | $s^nF(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$ | |
| $t^nf(t)$ | $(-1)^nF^{(n)}(s)$ | |