MATH 310C
SPRING 2007
SAMPLE EXAM I — SOLUTIONS

1. (a) Choose $x \in (A \cup C) - B$. Then $x \in A \cup C$ but $x \notin B$. Since $x \in A \cup C$, $x \in A$ or $x \in C$. If $x \in A$, then we have that $x \in A$ but $x \notin B$, which means that $x \in A - B$, which is a subset of $(A - B) \cup C$. If $x \notin A$, then x must be in C, which means that $x \in (A - B) \cup C$. Thus, every element of $(A \cup C) - B$ is also in $(A - B) \cup C$, which means that $(A \cup C) - B \subseteq (A - B) \cup C$.

   (b) As long as $B \cap C$ is non-empty, equality will not hold. One example is $A = \{1, 2, 3\}$, $B = \{1, 2\}$, and $C = \{2, 3, 4\}$. Then, $(A \cup C) - B = \{3, 4\}$ but $(A - B) \cup C = \{2, 3, 4\}$.

2. (a) The negation is: There exists an $x \in \mathbb{R}$ such that, for all $y \in \mathbb{R}$, $x + y \notin \mathbb{Z}$.

   (b) The negation is: For all $x \in \mathbb{Z}$, there exists $y \in \mathbb{Z}$ such that $x > y$ and $\frac{x^2}{y} \notin \mathbb{N}$.

3. By hypothesis, a is divisible by 3. Suppose that $a + b$ is also divisible by 3. Then $a + b = 3k$ for some integer k and $a = 3k'$ for some integer k'. So, $3k = a + b = 3k' + b$, which implies that $b = 3(k - k')$. The integers are closed under subtraction, which means that $k - k'$ is also an integer. Thus, b is divisible by 3. We've shown that, if $a + b$ is divisible by 3, then b must also be divisible by 3. Thus, the contrapositive is true: if b is not divisible by 3, then $a + b$ is not divisible by 3.

4. Base case: If $n = 1$, then $(1 + x)^n = 1 + x$ and $1 + 1 \cdot x = 1 + x$ and thus the conclusion is true.

   Induction step: Suppose that $(1+x)^k \geq 1+kx$ for some natural number k. We want to show that $(1+x)^{k+1} \geq 1+(k+1)x$. We start with the left-hand side: $(1+x)^{k+1} = (1+x)^k(1+x) \geq (1+kx)(1+x)$, by the induction hypothesis. We now have that $(1+x)^{k+1} \geq 1+kx+x+kx^2$, which in turn is greater than or equal to $1 + kx + x$, since $kx^2 \geq 0$. Thus, $(1 + x)^{k+1} \geq 1 + (k + 1)x$, which completes the induction step.

   We've shown that $(1 + x)^n \geq 1 + nx$ for all natural numbers n.

5. <u>$f(x)$ is injective</u>: Suppose $f(x_1) = f(x_2)$. Then

   $$\frac{x_1 + 1}{x_1 - 1} = \frac{x_2 + 1}{x_2 - 1}.$$

   If we multiply both sides by $(x_1 - 1)(x_2 - 1)$ to eliminate the denominators, we get:

   $$(x_2 - 1)(x_1 + 1) = (x_2 + 1)(x_1 - 1).$$

   Multiply out and combine like terms to see that $x_1 = x_2$.

   <u>$f(x)$ is not onto</u>: There is no x such that $f(x) = 1$. If such an x did exist, then

   $$\frac{x + 1}{x - 1} = 1.$$

   This would mean that $x+1 = x-1$, which would imply that $1 = -1$. This is a contradiction. Thus, there is an element of the target, $1 \in \mathbb{R}$, that is not in the image of f.

   Since $f(x)$ is not onto, $f(x)$ is not a bijection.