MATH 310C
SPRING 2007
SAMPLE EXAM II SOLUTIONS

1. (a) FALSE. Here is a counterexample: $f(x) = x$ and $g(x) = -x$ are both bijections. (Why?) But $h(x)$ is not. (Why not?)

   (b) TRUE.

   <u>Proof:</u> Suppose f and g are strictly increasing and that x_1 and x_2 are real numbers such that $x_1 < x_2$. Since f is strictly increasing, $x_1 < x_2$ implies that $f(x_1) < f(x_2)$. Similarly, $g(x_1) < g(x_2)$. This means that $f(x_1) + g(x_1) < f(x_2) + g(x_2)$ and by the definition of $h(x)$, $h(x_1) < h(x_2)$. Thus, h is strictly increasing.

2. (a) Since $d|a$ and $d|b$, there exist integers k and k' such that $a = dk$ and $b = dk'$. So, $\frac{a}{d} = k$ and $\frac{b}{d} = k'$, which are both integers.

   (b) Let k and k' be as defined in part (a). Then $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = \gcd(k, k')$. Let $n = \gcd(k, k')$. Then $n|k$ and $n|k'$. This means that there exist integers k_1 and k_1' such that $k = nk_1$ and $k' = nk_1'$. But then $a = dk = dnk_1$ and $b = dk' = dnk_1'$, which means that dn is a common divisor of a and b. If $n > 1$, then $dn > d$, which contradicts the fact that d is the greatest common divisor of a and b. So, n must be 1.

3. (a) Note that g is essentially the same function as f. The only difference is the target. So, g is injective because f is injective and g is surjective by the definition of $f(A)$. (The target of g is precisely those elements that get "hit" by $f(x)$ and therefore by $g(x)$.)

   (b) i. If $|A| = n$ for some $n \in \mathbb{N}$, then there exists a bijection $F : A \to [n]$. The function g from part (a) is a bijection from A onto $f(A)$. As such, g has an inverse, $g^{-1} : f(A) \to A$. Since the composition of two bijections is a bijection, $F \circ g^{-1}$ is a bijection from $f(A)$ onto $[n]$. Thus, $|f(A)| = n = |A|$.

   ii. If A is countable, then there exists a bijection from A onto $\mathbb{N}$. Again, using the bijection g from part (a), g^{-1} is a bijection from $f(A)$ onto A. So, $F \circ g^{-1}$ is a bijection from $f(A)$ onto $\mathbb{N}$. Thus, $f(A)$ is also countable.

   iii. If $f(A)$ is countable, then there exists a bijection F from $f(A)$ onto $\mathbb{N}$. Again, with $g : A \to f(A)$ the bijection from part (a), the function $F \circ g$ is a bijection from A onto $\mathbb{N}$. This means that A is countable. This shows that, if $f(A)$ countable, then A is countable. The converse is therefore true: if A is uncountable, then $f(A)$ is uncountable.

4. (a) By the Division Algorithm, there exist unique k and r such that $Q = 4k + r$ and $r = 0, 1, 2,$ or 3. Since Q is odd, r cannot be 0 or 2. So, r must be 1 or 3. By the uniqueness of r, Q is either of the form $4k + 1$ or $4k + 3$, but not both.

   (b) <u>Basis step:</u> $n = 1$: $a_1 \in S \Rightarrow a_1 \in S$ ✓

   <u>Induction step:</u> Suppose that, if $a_1, a_2, ..., a_i \in S$, then $a_1a_2...a_i \in S$ and consider the product $a_1a_2...a_ia_{i+1}$, where $a_1, a_2, ..., a_i, a_{i+1} \in S$. By induction, $a_1a_2...a_i = 4k + 1$ for some integer k and $a_{i+1} = 4k' + 1$ for some integer k'. Then,

$$a_1a_2...a_ia_{i+1} = (4k + 1)(4k' + 1) = 16kk' + 4k + 4k' + 1 = 4(4kk' + k + k') + 1 \in S.$$

(c) First, note that Q is odd. (Why?) So all of the prime factors of Q are odd. (Why?) If all of the prime factors of Q were of the form $4k + 1$, then Q would also be of the form $4k + 1$. (Why?) So, at least one of the prime factors of Q is of the form $4k + 3$.

(d) TRUE! The proof is extra credit.