Line Integral Practice

Scalar Function Line Integrals with Respect to Arc Length

For each example below compute, $\int_C f(x,y)\,ds$ or $\int_C f(x,y,z)\,ds$ as appropriate.

Problems:

1. C is the line segment from $(1,3)$ to $(5,-2)$, compute $\int_C x - y\,ds$
2. C is the line segment from $(3,4,0)$ to $(1,4,2)$, compute $\int_C z + y^2\,ds$.
3. C is the curve from $y = x^2$ from $(0,0)$ to $(3,9)$, compute $\int_C 3x\,ds$.
4. C is the upper half of the circle of radius 2 from $(2,0)$ to $(-2,0)$, compute $\int_C y\,ds$.

Solutions:

1. (a) Parameterization: $x = 1 + 4t$, $y = 3 - 5t$, $0 \leq t \leq 1$

   (b) Integration: $\int_C x - y\,ds = \int_0^1 [(1+4t) - (3-5t)]\sqrt{4^2+5^2}\,dt = \sqrt{41}\int_0^1 -2 + 9t\,dt = \frac{5}{2}\sqrt{41}.$
2. (a) Parameterization: $x = 3 - 2t$, $y = 4$, $z = 2t$, $0 \leq t \leq 1$

   (b) Integration: $\int_C z + y^2\,ds = \int_0^1 [2t + 16]\sqrt{2^2+0^2+2^2}\,dt = \sqrt{8}\int_0^1 2t + 16\,dt = 17\sqrt{8}.$
3. (a) Parameterization: $x = t$, $y = t^2$, $0 \leq t \leq 3$

   (b) Integration: $\int_C 3x\,ds = \int_0^3 3t\sqrt{1^2 + (2t)^2}\,dt = \frac{3}{8}\int_1^{37} \sqrt{u}\,du = \frac{1}{4}(37^{3/2} - 1).$
4. (a) Parameterization: $x = 2\cos(t)$, $y = 2\sin(t)$, $0 \leq t \leq \pi$

   (b) Integration: $\int_C y\,ds = \int_0^\pi 2\sin(t)\sqrt{(-2\sin(t))^2 + (2\cos(t))^2}\,dt = 4\int_0^\pi \sin(t)\,dt = 8.$

Vector Function Line Integrals

For each example below compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Problems:

1. C is the line segment from $(2,3)$ to $(0,3)$ and $\mathbf{F} = \langle x, -y \rangle$, compute $\int_C \mathbf{F} \cdot d\mathbf{r}$
2. C is the line segment from $(5,0,2)$ to $(5,3,4)$ and $\mathbf{F} = \langle z, -y, x \rangle$, compute $\int_C \mathbf{F} \cdot d\mathbf{r}$
3. C is the curve from $y = e^x$ from $(2, e^2)$ to $(0,1)$ and $\mathbf{F} = \langle x^2, -y \rangle$, compute $\int_C \mathbf{F} \cdot d\mathbf{r}$
4. C is the part of the circle of radius 3 in the first quadrant from $(3,0)$ to $(0,3)$ and $\mathbf{F} = \langle 1, -y \rangle$, compute $\int_C \mathbf{F} \cdot d\mathbf{r}$
5. C is the part of the curve $x = \cos(y)$ from $(1, 2\pi)$ to $(1,0)$ and $\mathbf{F} = \langle y, 2x \rangle$, compute $\int_C \mathbf{F} \cdot d\mathbf{r}$

Solutions:

1. (a) Parameterization: $x = 2 - 2t$, $y = 3$, $0 \le t \le 1$
   (b) Integration: $\int_C \langle x, -y \rangle \cdot d\mathbf{r} = \int_0^1 \langle 2-2t, -3 \rangle \cdot \langle -2, 0 \rangle \, dt = \int_0^1 -4 + 4t \, dt = -2.$
2. (a) Parameterization: $x = 5$, $y = 3t$, $z = 2 + 2t$, $0 \le t \le 1$
   (b) Integration: $\int_C \langle z, -y, x \rangle \cdot d\mathbf{r} = \int_0^1 \langle 2+2t, -3t, 5 \rangle \cdot \langle 0, 3, 2 \rangle \, dt = \int_0^1 -9t + 10 \, dt = \frac{11}{2}.$
3. (a) Parameterization: $x = t$, $y = e^t$, $0 \le t \le 2$. NOTE: The parameterization I've given has the wrong orientation. So we can either change the parameterization (change all t's to $-t$'s), or just note that this is the parameterization of $-C$ and change the sign of what we get. That is what I will do below.
   (b) Integration: $\int_C \langle x^2, -y \rangle \cdot d\mathbf{r} = -\int_{-C} \langle x^2, -y \rangle \cdot d\mathbf{r} = -\int_0^2 \langle t^2, -e^t \rangle \cdot \langle 1, e^t \rangle \, dt = -\int_0^2 t^2 - e^{2t} \, dt = -\left(\frac{8}{3} - \frac{1}{2}e^4 + \frac{1}{2} \right) = -\frac{19}{6} + \frac{1}{2}e^4.$
4. (a) Parameterization: $x = 3\cos(t)$, $y = 3\sin(t)$, $0 \le t \le \frac{\pi}{2}$. NOTE: This parameterization has the correct orientation.
   (b) Integration: $\int_C \langle 1, -y \rangle \cdot d\mathbf{r} = \int_0^{\pi/2} \langle 1, -3\sin(t) \rangle \cdot \langle -3\sin(t), 3\cos(t) \rangle \, dt = \int_0^{\pi/2} -3\sin(t) - 9\sin(t)\cos(t) \, dt = \cdots = -\frac{15}{2}.$
5. (a) Parameterization: $x = \cos(t)$, $y = t$, $0 \le t \le 2\pi$. NOTE: The parameterization I've given has the wrong orientation. So we can either change the parameterization (change all t's to $-t$'s), or just note that this is the parameterization of $-C$ and change the sign of what we get. That is what I will do below.
   (b) Integration: $\int_C \langle y, 2x \rangle \cdot d\mathbf{r} = -\int_0^{2\pi} \langle t, 2\cos(t) \rangle \cdot \langle -\sin(t), 1 \rangle \, dt = -\int_0^{2\pi} -t\sin(t) + 2\cos(t) \, dt = \cdots = -2\pi.$