# Line Integral Application List

## Scalar Function Line Integrals with Respect to Arc Length:

$$
\begin{aligned}
&\int_C 1\,ds && = \text{arc length of curve } C.\\
&\int_C f(x,y)\,ds && = \text{area above } C \text{ and below } f(x,y).\\
&\int_C \rho(x,y)\,ds && = \text{total mass corresponding to density } \rho(x,y) \text{ over } C.\\
&\frac{1}{\text{Total Mass}}\int_C x\rho(x,y)\,ds && = x\text{-coordinate of center of mass}\\
&\frac{1}{\text{Total Mass}}\int_C y\rho(x,y)\,ds && = y\text{-coordinate of center of mass}\\
&\frac{1}{\text{Arc Length}}\int_C f(x,y)\,ds && = \text{average value of } f(x,y) \text{ over } C.
\end{aligned}
$$

$$
\begin{aligned}
&\int_C \rho(x,y,z)\,ds && = \text{total mass corresponding to density } \rho(x,y,z) \text{ over } C.\\
&\frac{1}{\text{Total Mass}}\int_C x\rho(x,y,z)\,ds && = x\text{-coordinate of center of mass}\\
&\frac{1}{\text{Total Mass}}\int_C y\rho(x,y,z)\,ds && = y\text{-coordinate of center of mass}\\
&\frac{1}{\text{Total Mass}}\int_C z\rho(x,y,z)\,ds && = z\text{-coordinate of center of mass}\\
&\frac{1}{\text{Arc Length}}\int_C f(x,y,z)\,ds && = \text{average value of } f(x,y,z) \text{ over } C.
\end{aligned}
$$

**Vector Function Line Integrals**:

Recall from Math 126, that the component projection of the vector $\mathbf{a}$ onto the vector $\mathbf{b}$ is given by $\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$. In particular, if the vector $\mathbf{b}$ has unit length, then the component projection of $\mathbf{a}$ onto $\mathbf{b}$ is just given by $\mathbf{a}\cdot\mathbf{b}$. Assume we are traveling along a curve $\mathbf{r}(t)$. Recall from Math 126, at a particular point on this curve, we have $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} =$ the unit tangent vector $=$ the unit direction the curve is heading. Now assume we are given a vector field $\mathbf{F}$ and we are studying what is happening along the curve $\mathbf{r}(t)$.

$$\mathbf{F}\cdot\mathbf{T} = \text{ the projection of vector field at the point in the direction of the unit tangent.}$$

Note that:

- When $\mathbf{F}\cdot\mathbf{T}$ is positive, it tells us the vector field is pushing with the object moving along the curve.When $\mathbf{F}\cdot\mathbf{T}$ is negative, it tells us the vector field is pushing against the object moving the curve.
- If $\mathbf{F}$ is force (let's say in Newtons), then $\mathbf{F}\cdot\mathbf{T}$ is the component of force acting in the direction of the curve (still in Newtons). And if we integrate this component force, we will find the work done by the vector field on the object moving along the curve (which is 'adding up' a bunch of small force times distance computations and the units will be Newton-meters = Joules).
- If $\mathbf{v}$ is a velocity vector field (let's say in meters per second), then $\mathbf{v}\cdot\mathbf{T}$ is the component of velocity acting in the direction of the curve (still in meters per second). And if we integrate this component velocity, we will find what is called the **circulation** or **flow** of $\mathbf{v}$ along $C$. It gives a measure of the tendency of the fluid to move along $C$, positive means it flows in the same direction orientation as the curve and negative means it flows in the opposite orientation. (in meters/second * meters)

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\,|\mathbf{r}'(t)|dt = \int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_C \mathbf{F}\cdot\,d\mathbf{r}$$

If we write $\mathbf{F} = \langle P(x,y), Q(x,y)\rangle$, then this can also be written as

$$\int_C \mathbf{F}\cdot\,d\mathbf{r} = \int_C \langle P,Q\rangle\cdot\langle x'(t),y'(t),z'(t)\rangle dt = \int_C P(x(t),y(t))x'(t)dt + Q(x(t),y(t))y'(t)dt = \int_C Pdx+Qdy.$$

*Aside* (not in this homework, yet we will come back to this in 16.5): In 2D, when studying velocity vector fields, we are often more interest in the flow that is 'outward' across a curve rather than in the direction of the curve. In such cases, we often consider the outward pointing normal vector $\mathbf{n}(t) = \langle \frac{y'(t)}{|\mathbf{r}'(t)|}, -\frac{x'(t)}{|\mathbf{r}'(t)|}\rangle$. See page 1096 of your book for a picture of this vector, but you definitely should be able to quickly confirm that it is perpendicular to the tangent vector $\langle x'(t), y'(t)\rangle, so it must be pointing perpendicular to the curve.$

Thus, $\mathbf{F}\cdot\mathbf{n}$ would give the component projection of the velocity vector field perpendicular to the object moving along the curve in the outward direction, so if $\mathbf{F}$ pushes outwardly then the projection is positive and if it pushes inwardly the projection is negative. If we integrate this quantity, then we get a measure of the flow across the curve (the tendency of the fluid to flow outwardly across the curve). Hence we have

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \text{ the outward flow across the curve}$$

It can be rewritten many ways as well, but one common way is

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_C Pdy - Qdx.$$